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Bayes' Theorem & Probability Revision

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Learning Objectives and Outline

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Learning Objectives

- Recognize the importance of Bayes' Theorem within a Decision Analysis context
- Reproduce Bayes' within context of sensitivity/specificity/positive & negative predictive values

Medical Testing

Testing is done for:

- Screening (primary prevention)
- Diagnosis (secondary prevention)
- Monitor and guide treatment (tertiary prevention)
- Prognosis

Information for decision making

Clinicians have a variety of diagnostic information to guide their decision making

- Talking to patient (history, symptoms)
- Physically examining patient
- Screening (cervical cancer) + diagnostic tests (EKGs, Blood tests, X-rays)

Information for decision making

Obtaining information can be…

- RISKY
- EXPENSIVE
- ERROR PRONE
- ALL THREE

Role of Decision Analysis Methods

- Can be used to weigh the costs and benefits:
  - Test costs
  - Test accuracy
  - Health risks of testing

Example: Diagnostic Tests

What is the chance that a patient has a disease if a diagnostic test is positive or negative?

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Example: Diagnostic Tests

What is the chance that a patient has a disease if a diagnostic test is positive or negative?

In other words, what is the probability of disease conditional on the test result? (D+ | T+); (D+ | T-)

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Ways to get revised (posterior) probabilities

1. Bayes' theorem
2. 2x2 tables
3. Bayes' theorem via decision tree inversion

Ways to get revised (posterior) probabilities

1. Bayes' theorem
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1. Bayes' Theorem: Intuition

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Population of 200 indiviudals

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Disease (red) has 1% prevalence.

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A Screening Test is Available

Test +

Test -

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Test will detect 100% of positive cases.

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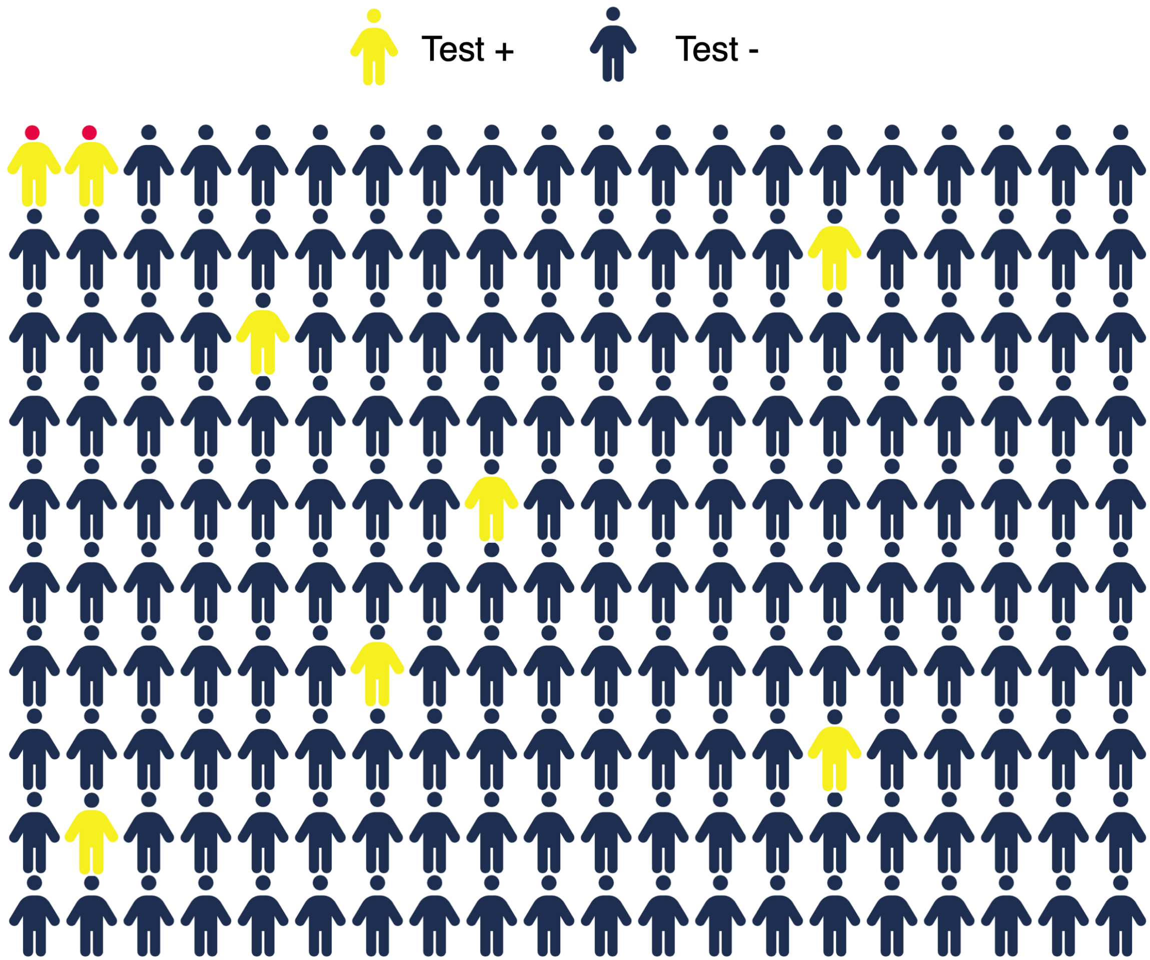
3% of disease-free individuals will test positive



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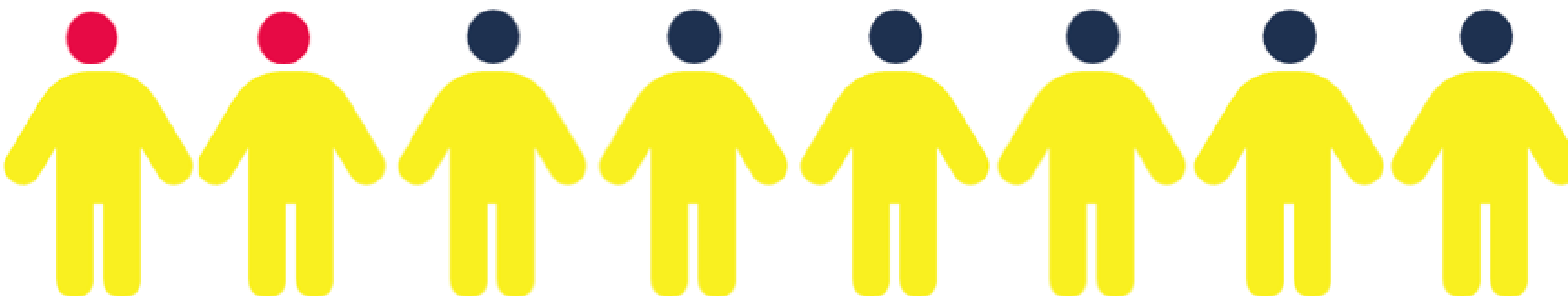
Eight Positive Tests



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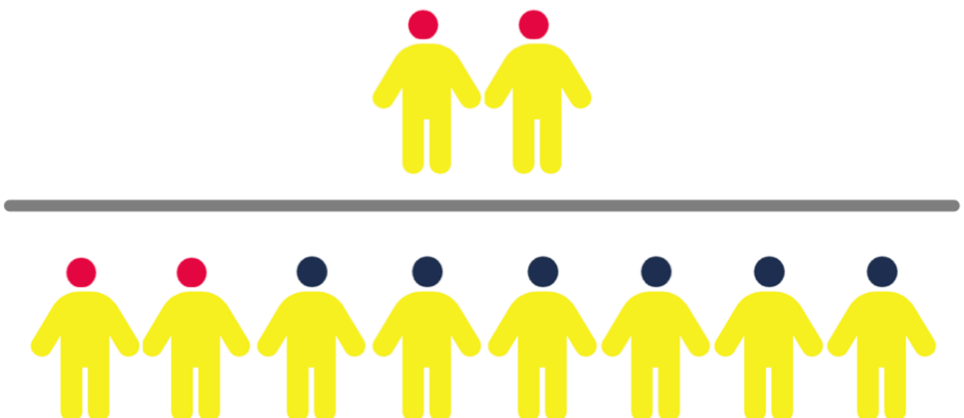
Probability of Disease Given Positive Test

100 *



= 25%

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Key Points

- While the test *always* detects a true case, it also returns false positives.
- The higher the number of false positives, the lower the probability that a given positive result is actually an individual with the disease.

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False Positives

- Could be the result of a poor test (e.g., 3% false positive rate).
- Or could be the result of a disease with very low prevalence in a large population, but a very good test!
  - In a population of 200 million, a test with a 0.01% false positive rate will yield 20,000 false positive readings!
  - If the disease prevalence is low (e.g., 0.001%, or 2,000 sick individuals), the probability that someone is sick if they test positive is <10%!

Bayes’ Theorem: Theory

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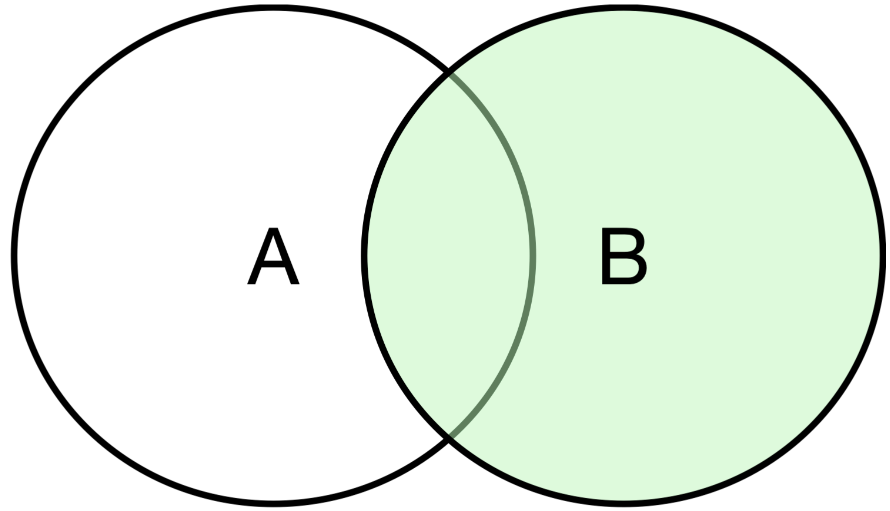
Probability of B

$$Pr(B)$$



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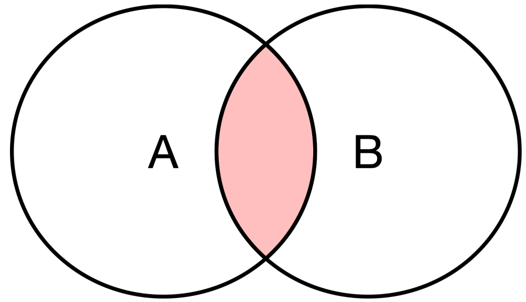
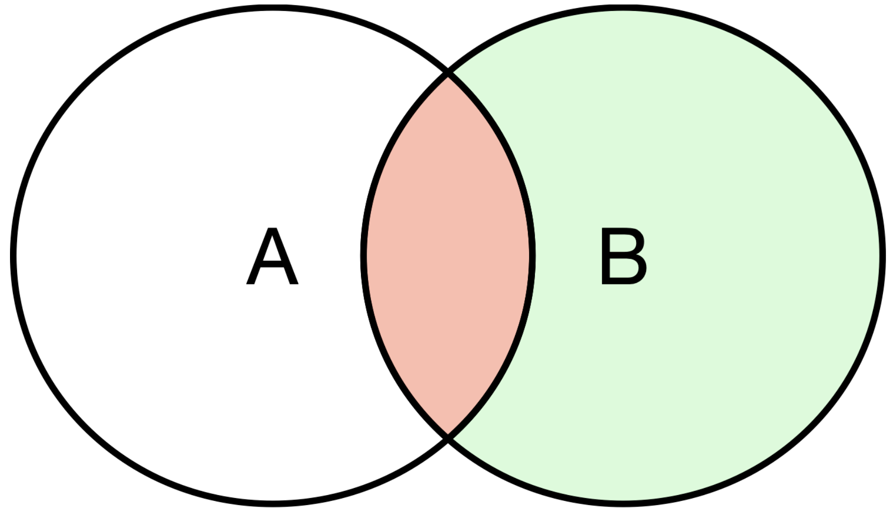
Probability of A and B

$$
\begin{aligned}
Pr(A\&B) &= Pr(A|B)Pr(B) \\
&= Pr(B|A)Pr(A)
\end{aligned}
$$

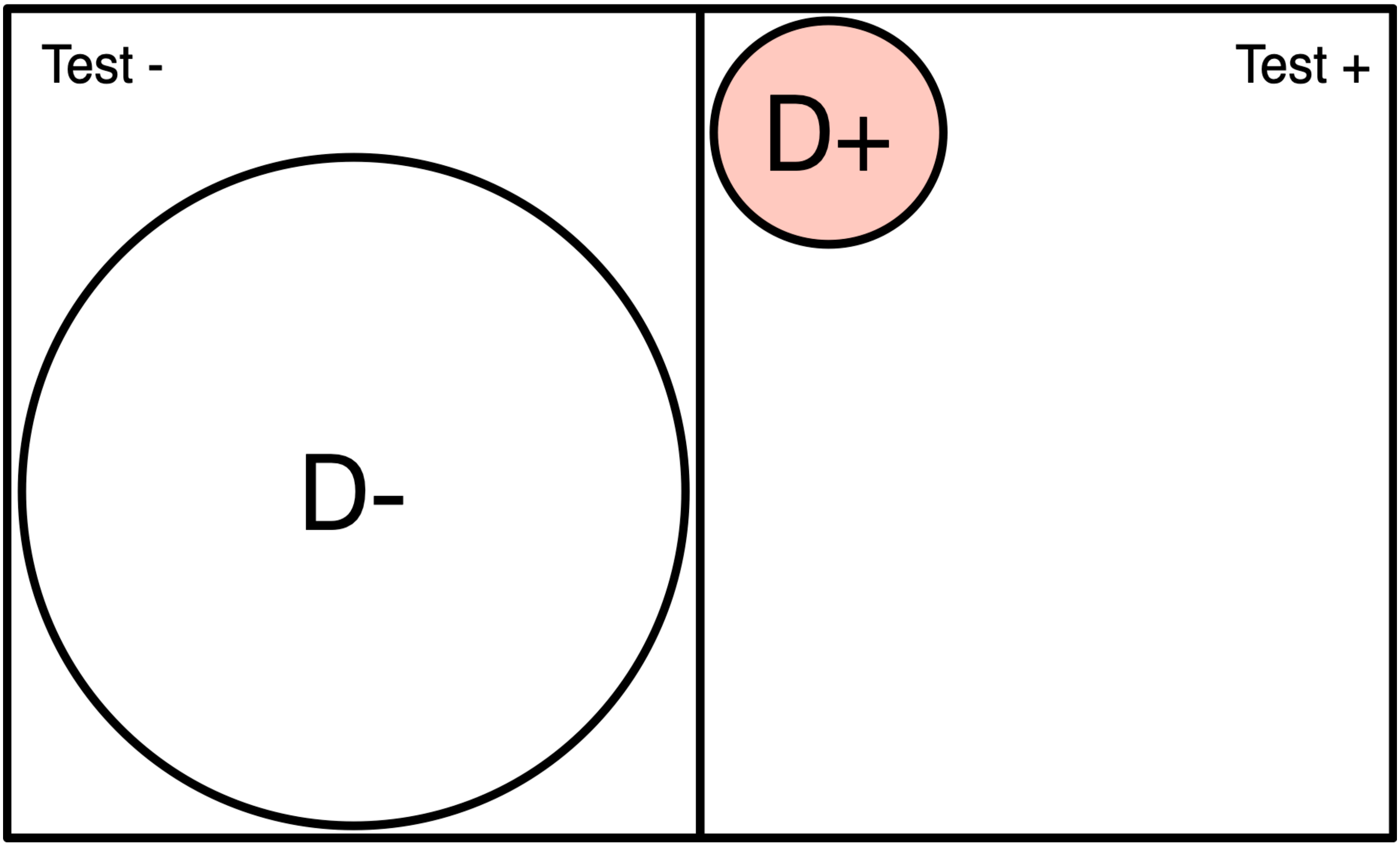


Probability of A Given B

$$Pr(A|B) = \frac{Pr(A\&B)}{Pr(B)}$$



A Perfect Test



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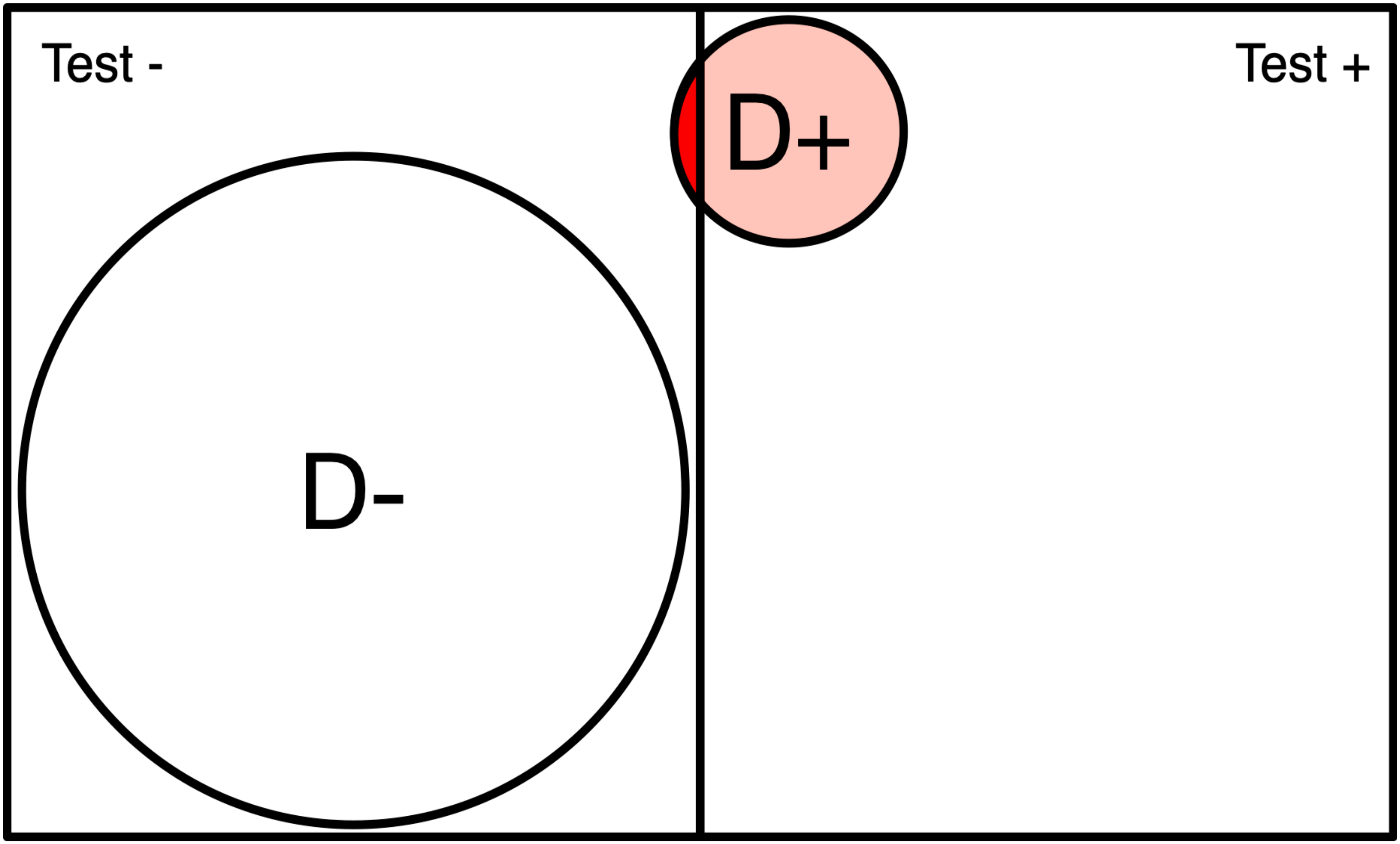
100% Specificity, <100% Sensitivity



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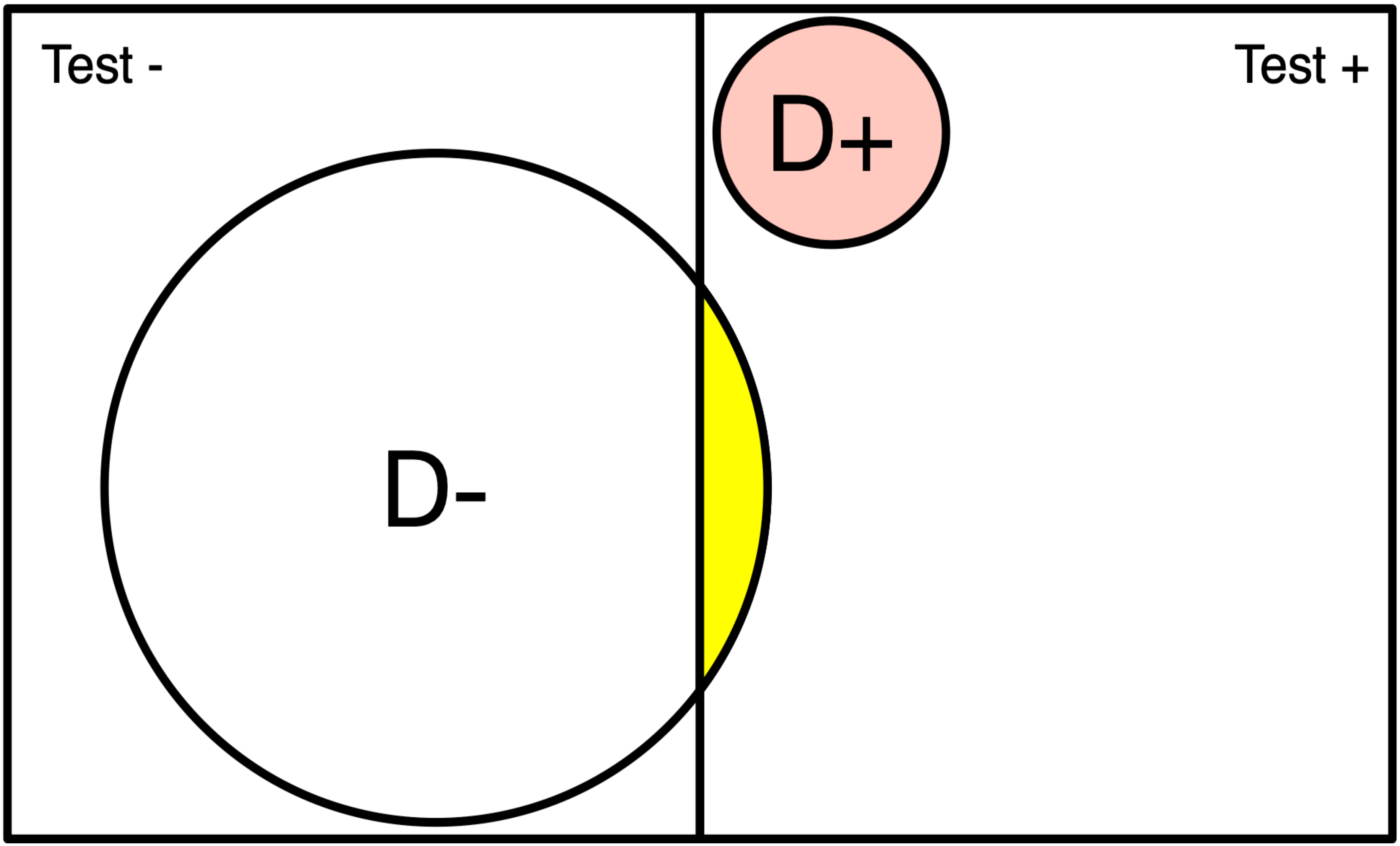
<100% Specificity, 100% Sensitivity



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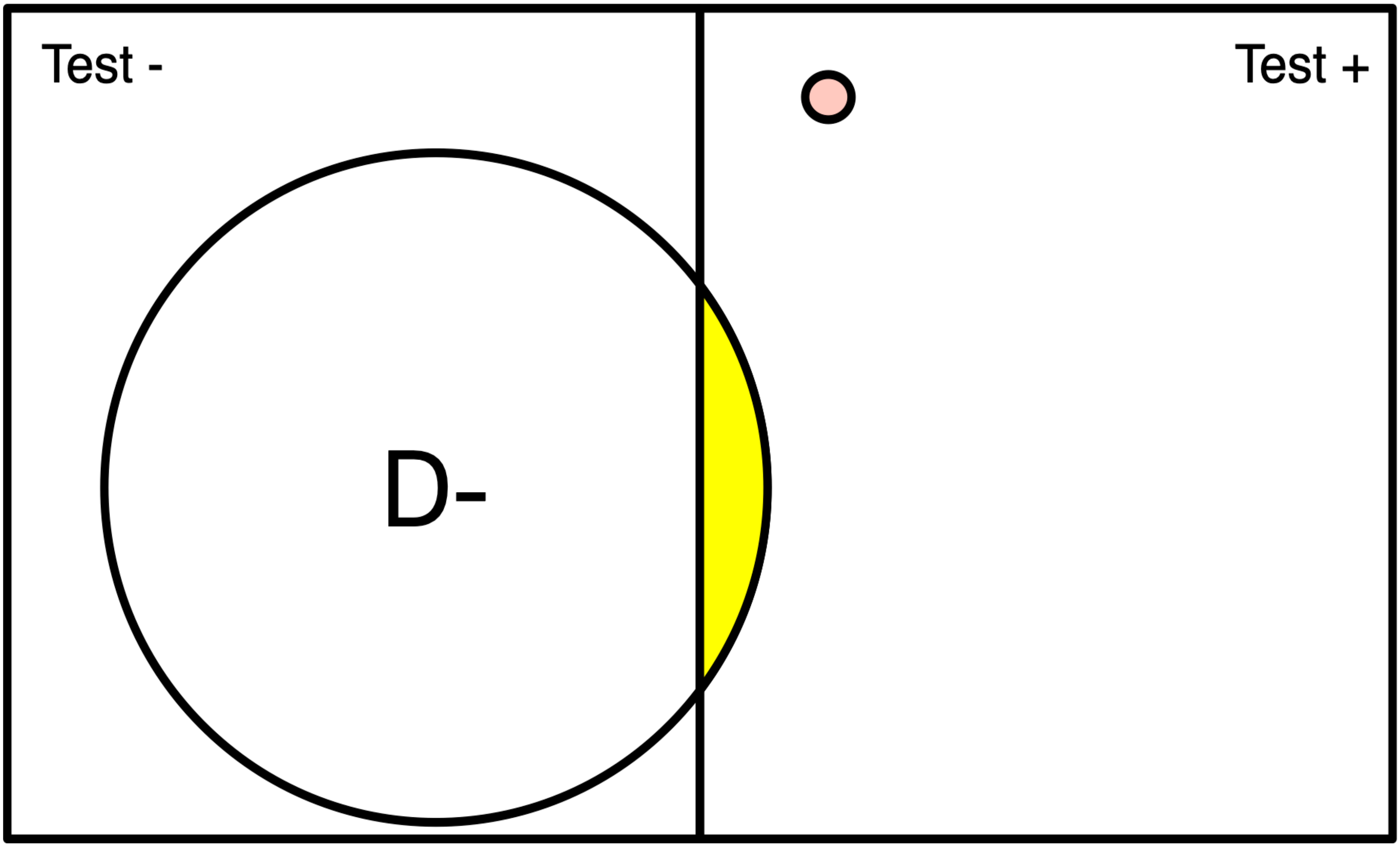
Let's Lower Disease Prevalence



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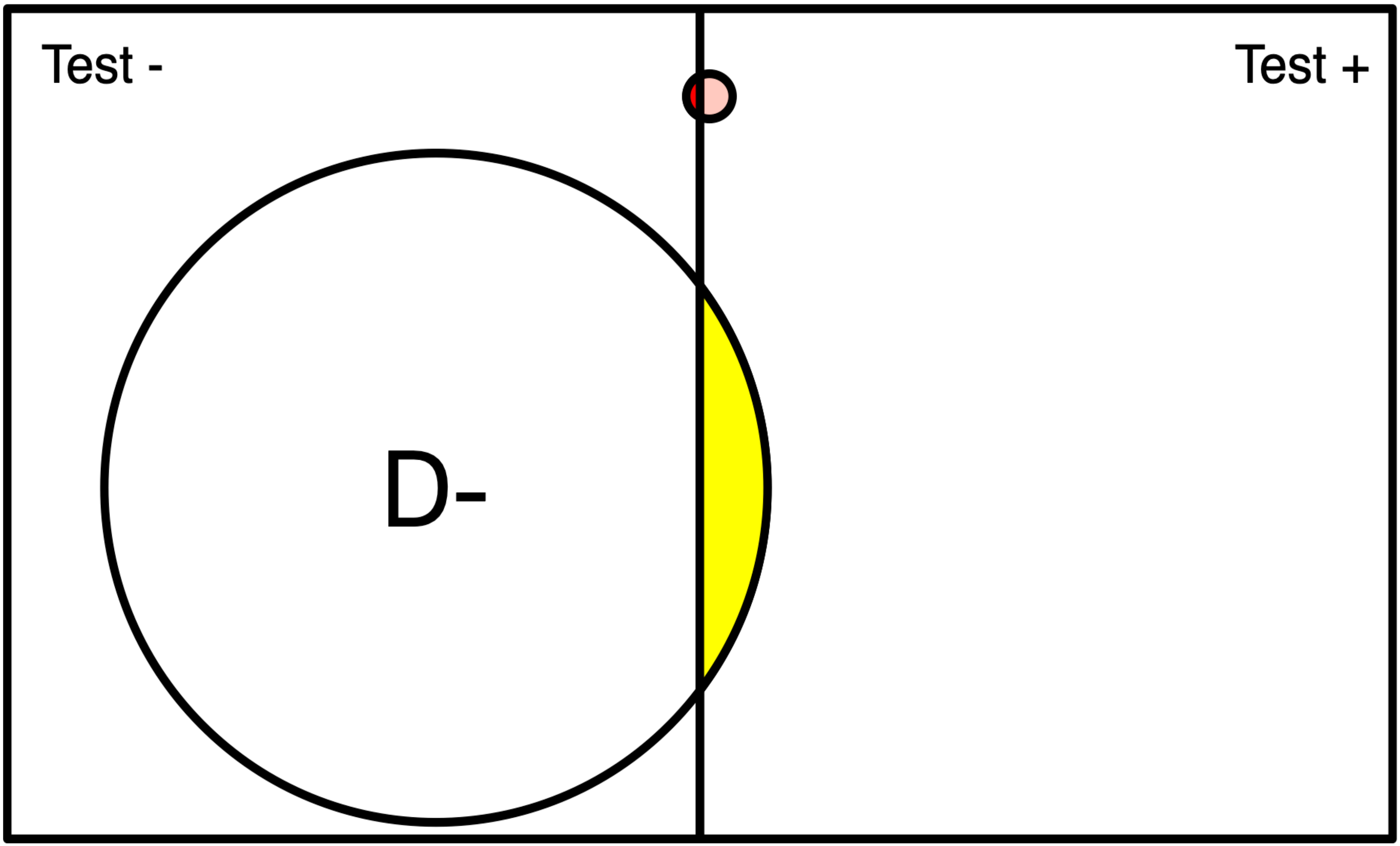
<100% Specificity, <100% Sensitivity



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What is the Probability of Disease Given A Positive Test?

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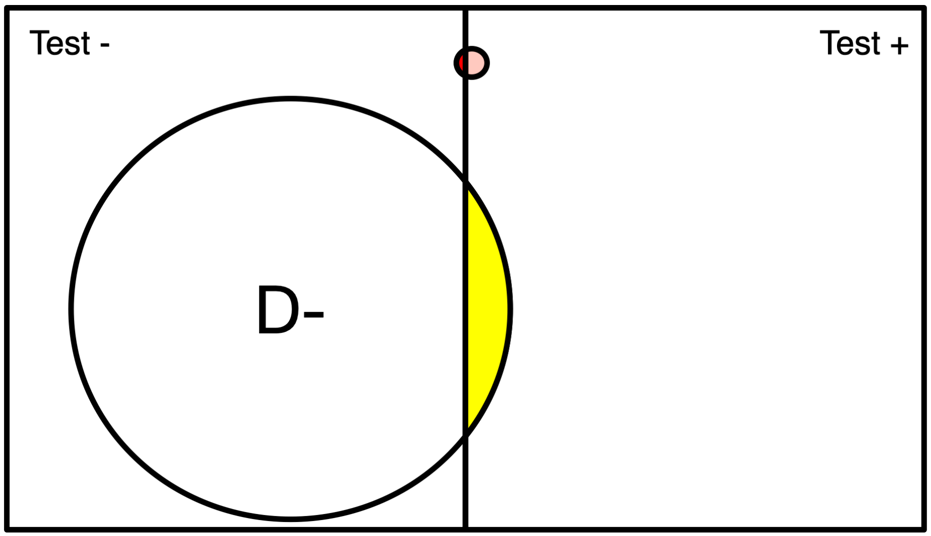
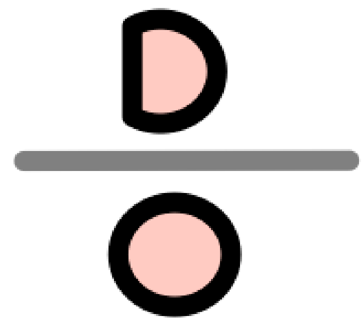
Probability of Testing Positive if Have Disease





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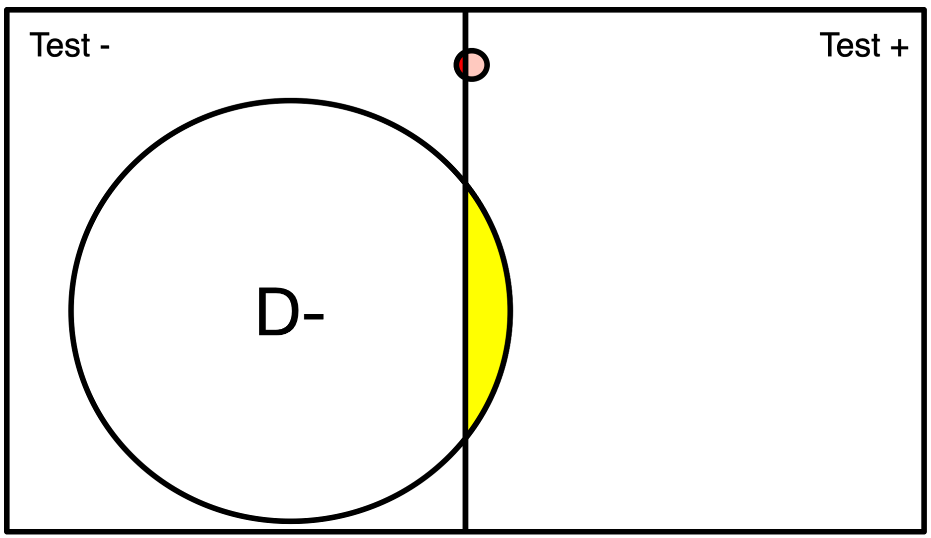
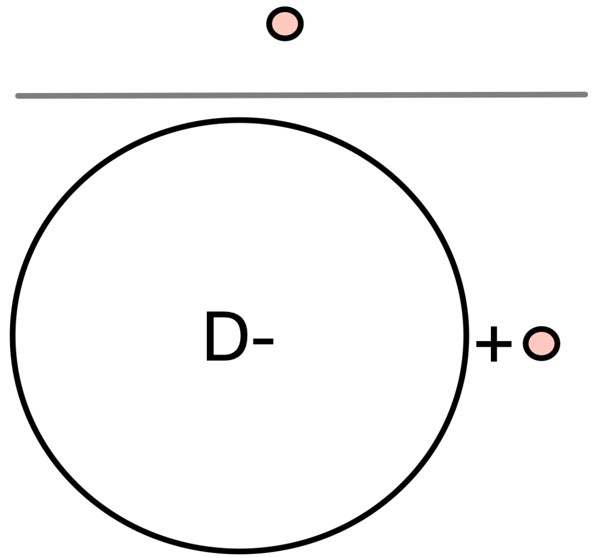
Probability of Having the Disease





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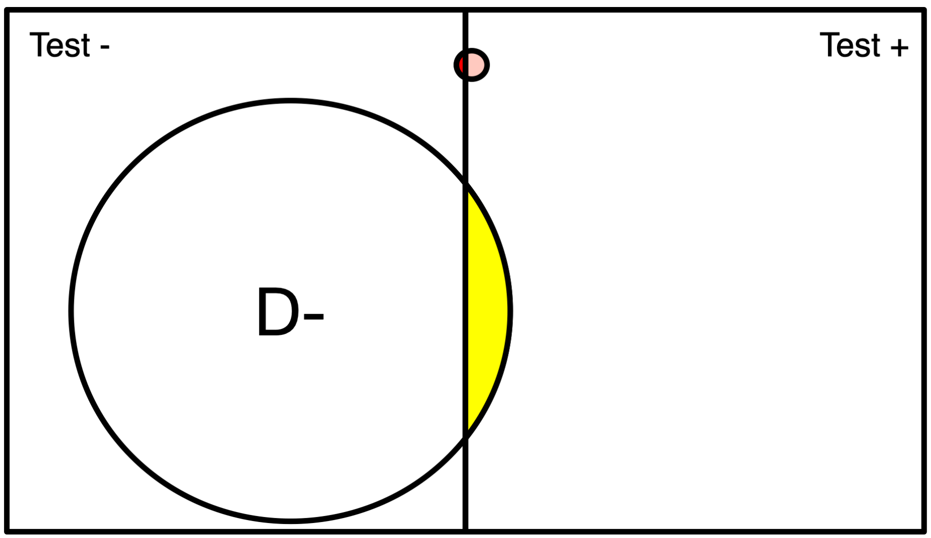
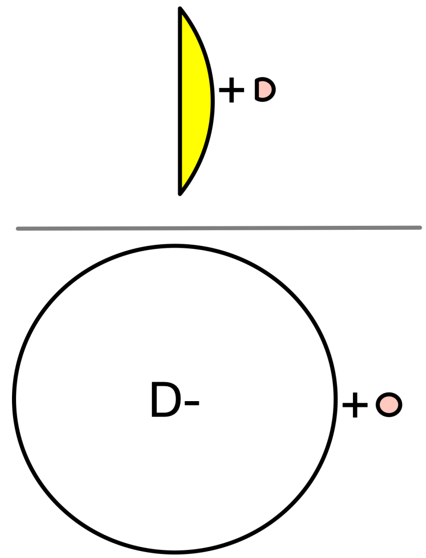
Probability of a Positive Test





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Bayes’ Theorem

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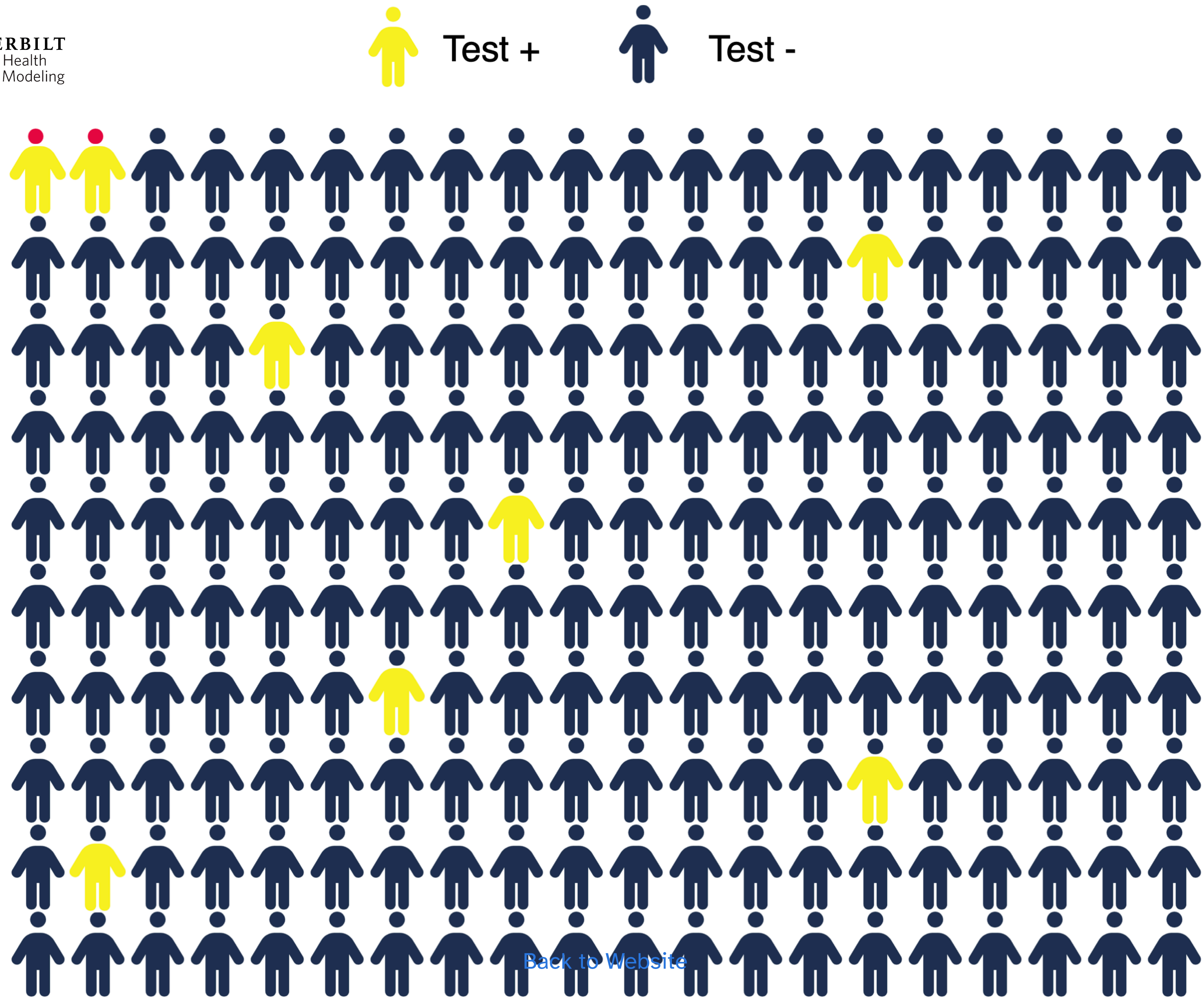

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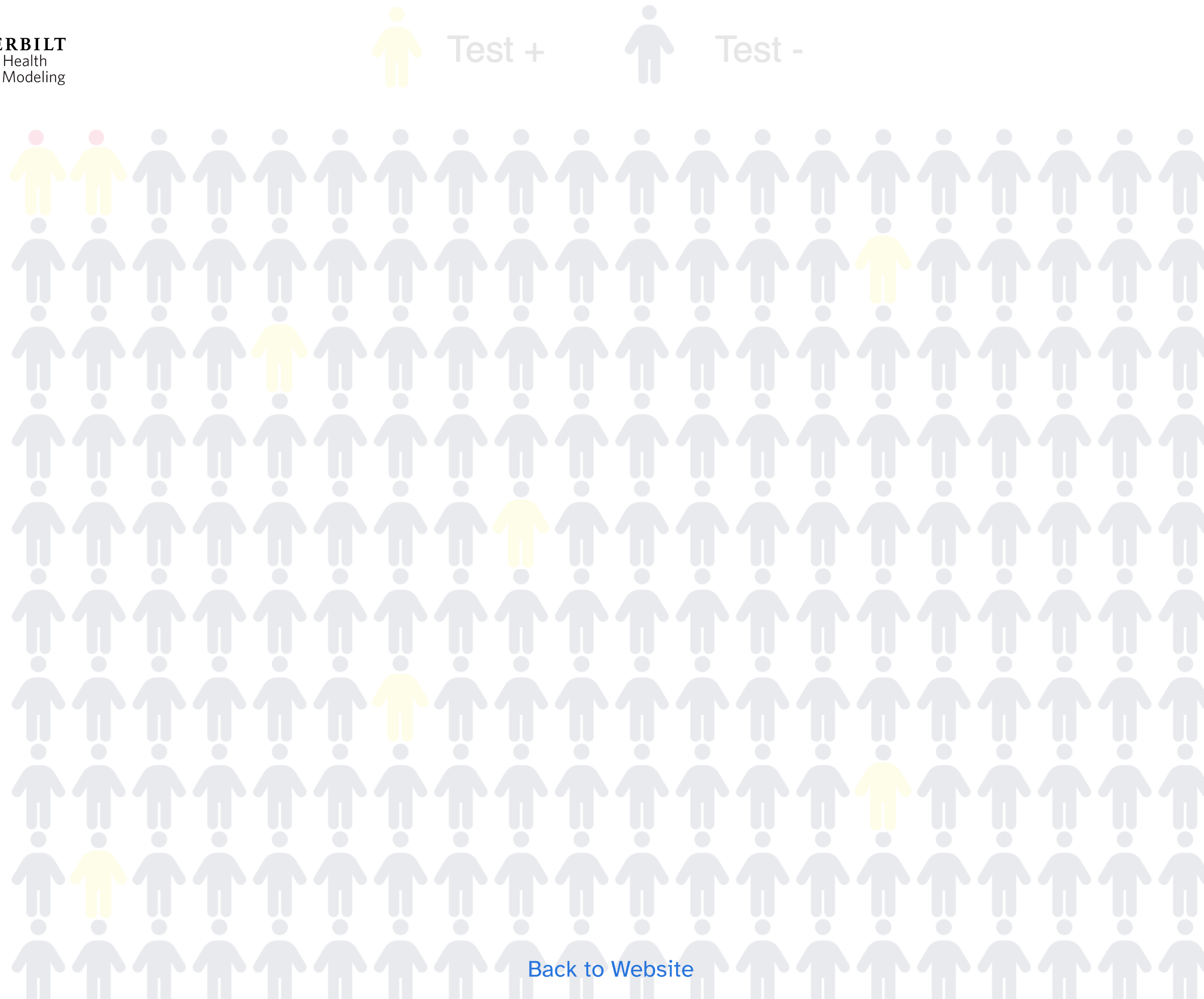

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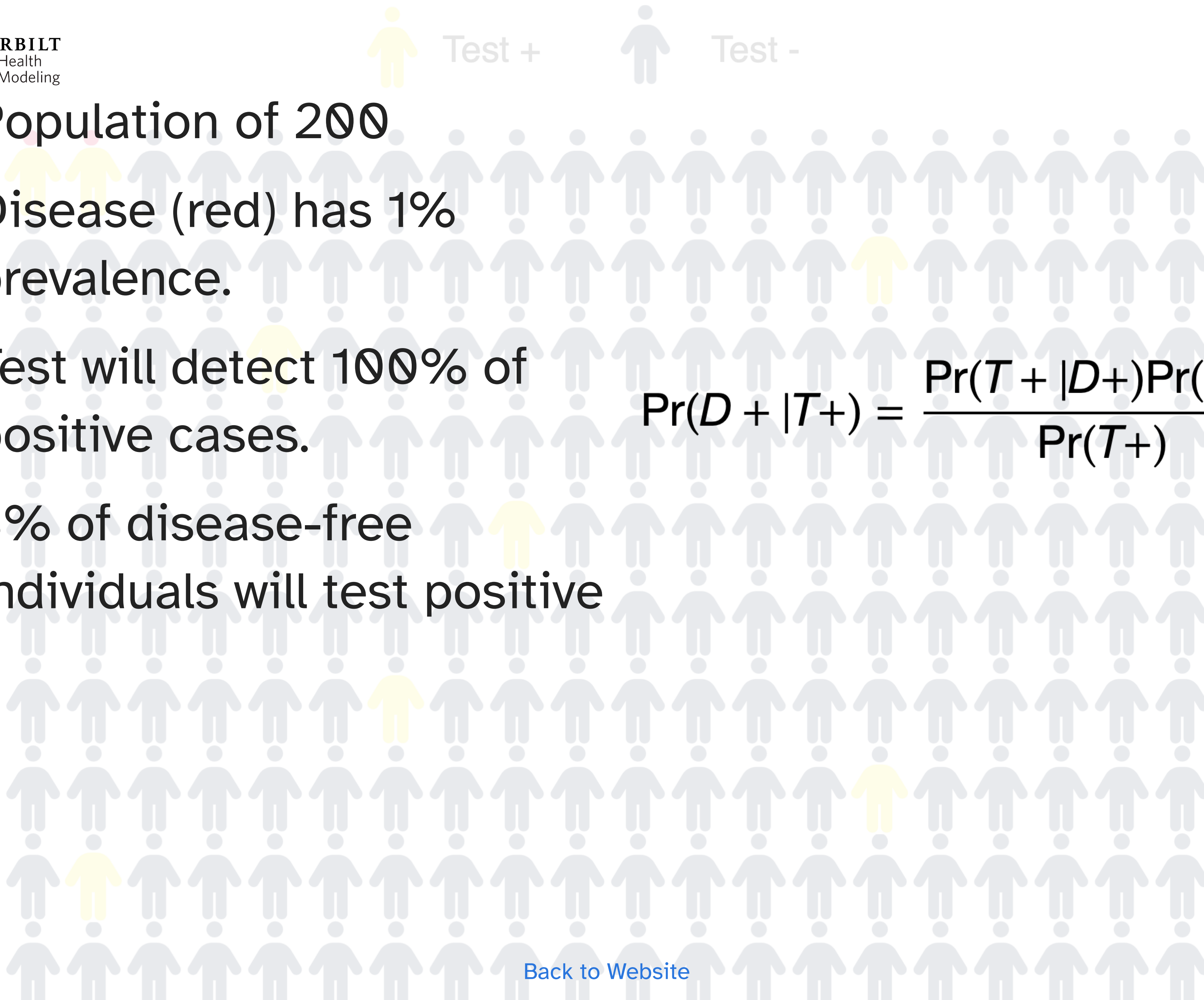

- Population of 200
- Disease (red) has 1% prevalence.
- Test will detect 100% of positive cases.
- 3% of disease-free individuals will test positive

$$\Pr(D+|T+) = \frac{\Pr(T+|D+)\Pr(D+)}{\Pr(T+)}$$

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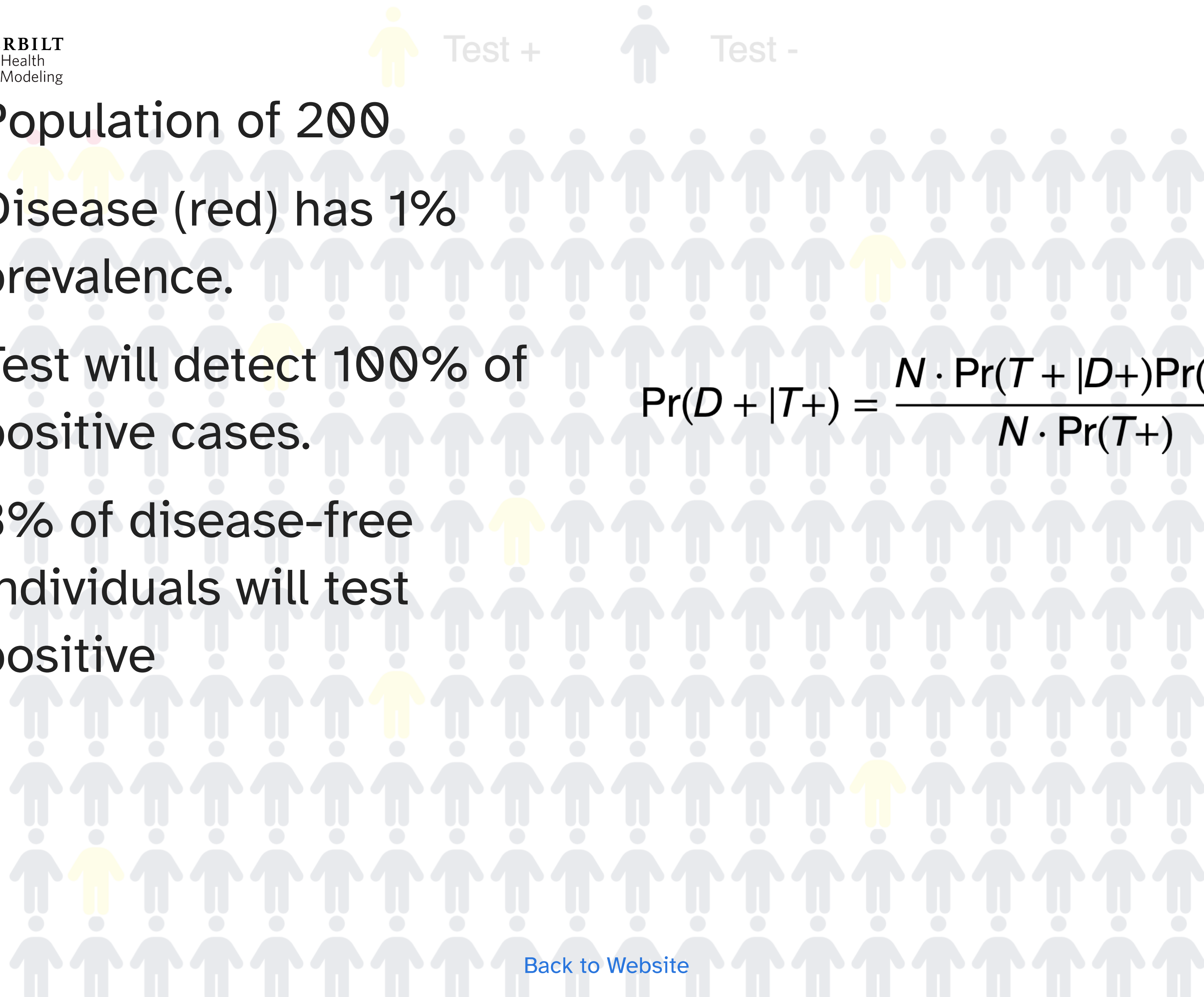

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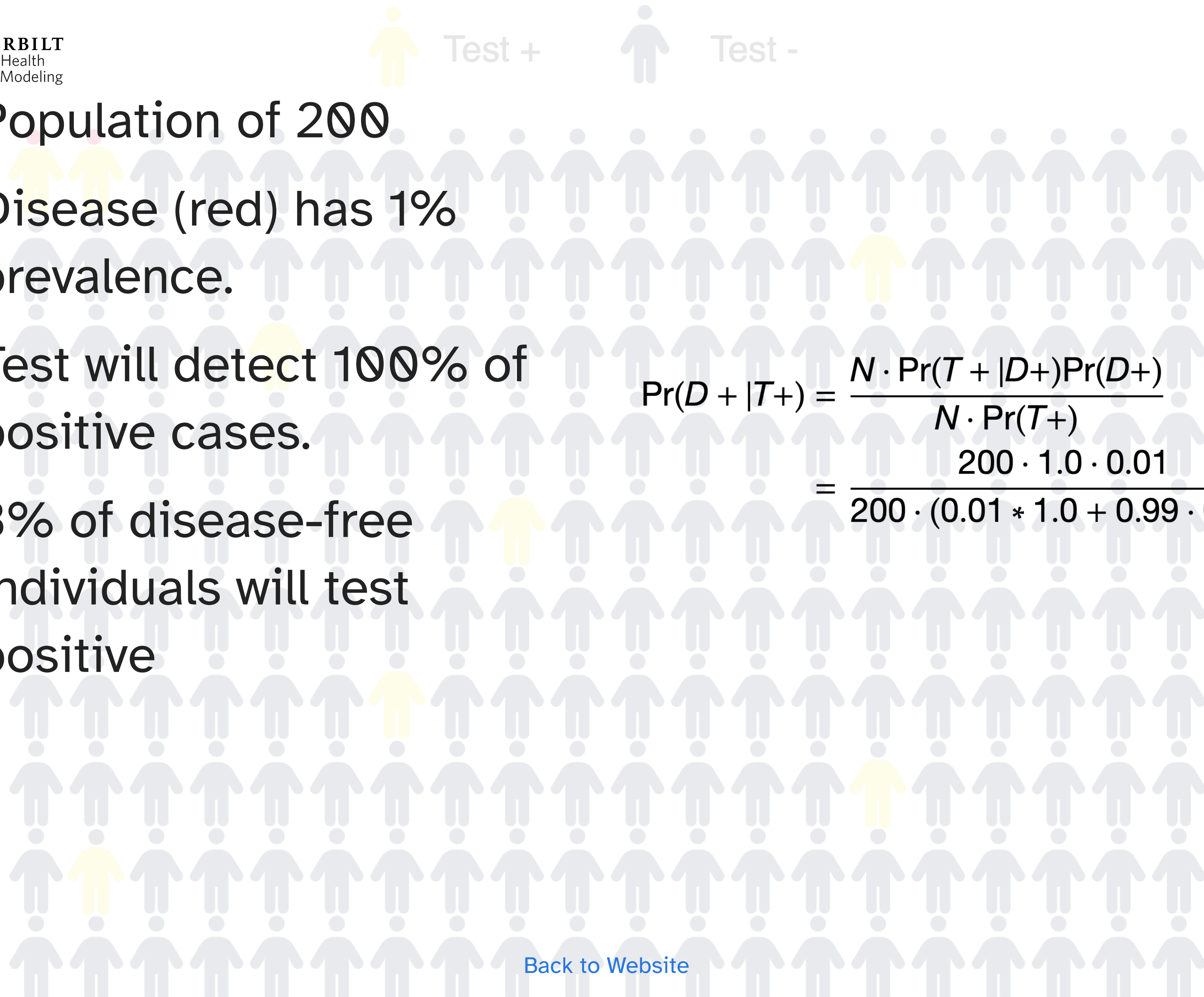

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$$\Pr(D+|T+) = \frac{N \cdot \Pr(T+|D+)\Pr(D+)}{N \cdot \Pr(T+)}$$

$$= \frac{200 \cdot 1.0 \cdot 0.01}{200 \cdot (0.01 * 1.0 + 0.99 \cdot 0.03)}$$

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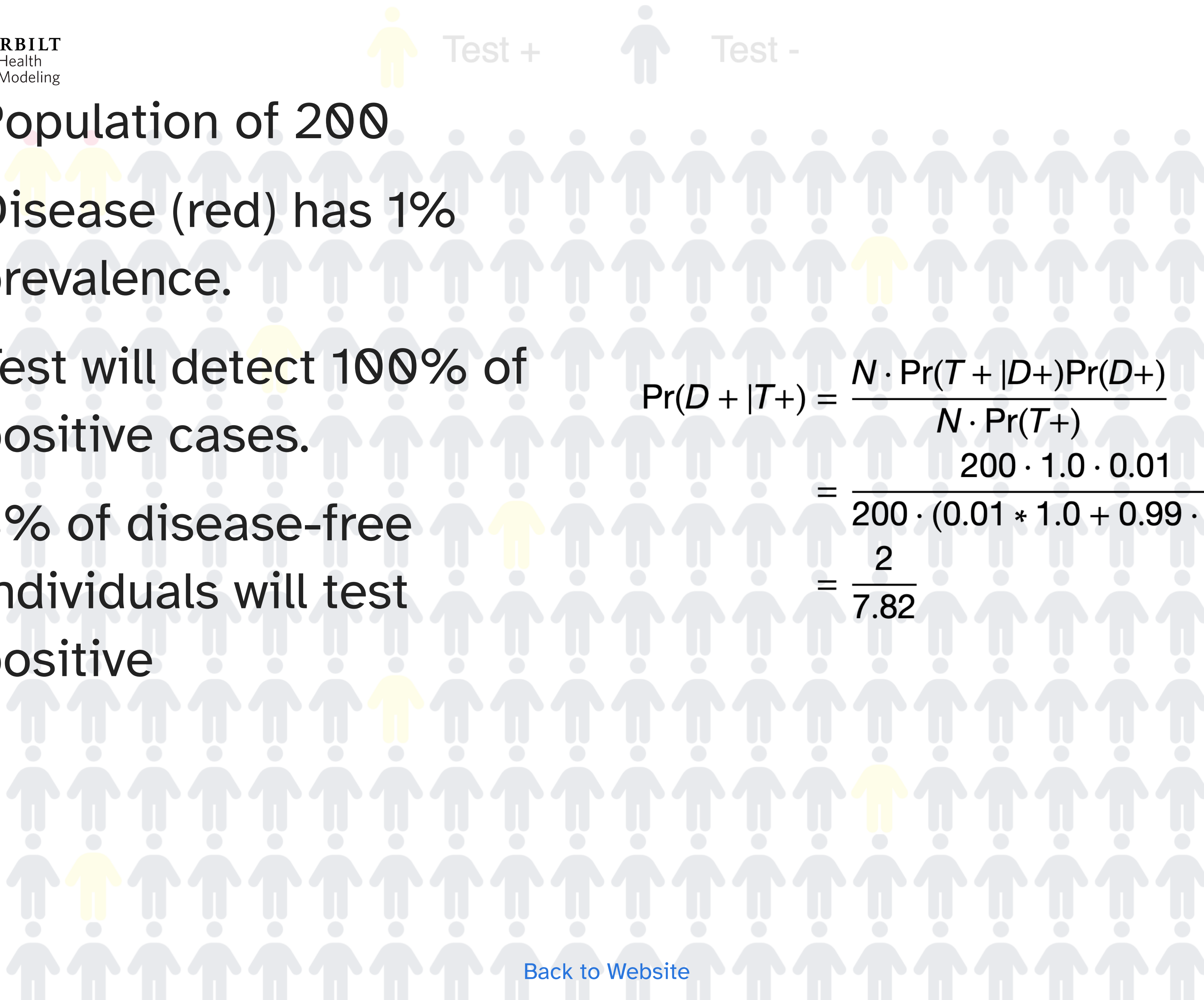

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$$\begin{aligned}
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&= \frac{200 \cdot 1.0 \cdot 0.01}{200 \cdot (0.01 \ast 1.0 + 0.99 \cdot 0.03)} \\
&= \frac{2}{7.82}
\end{aligned}$$

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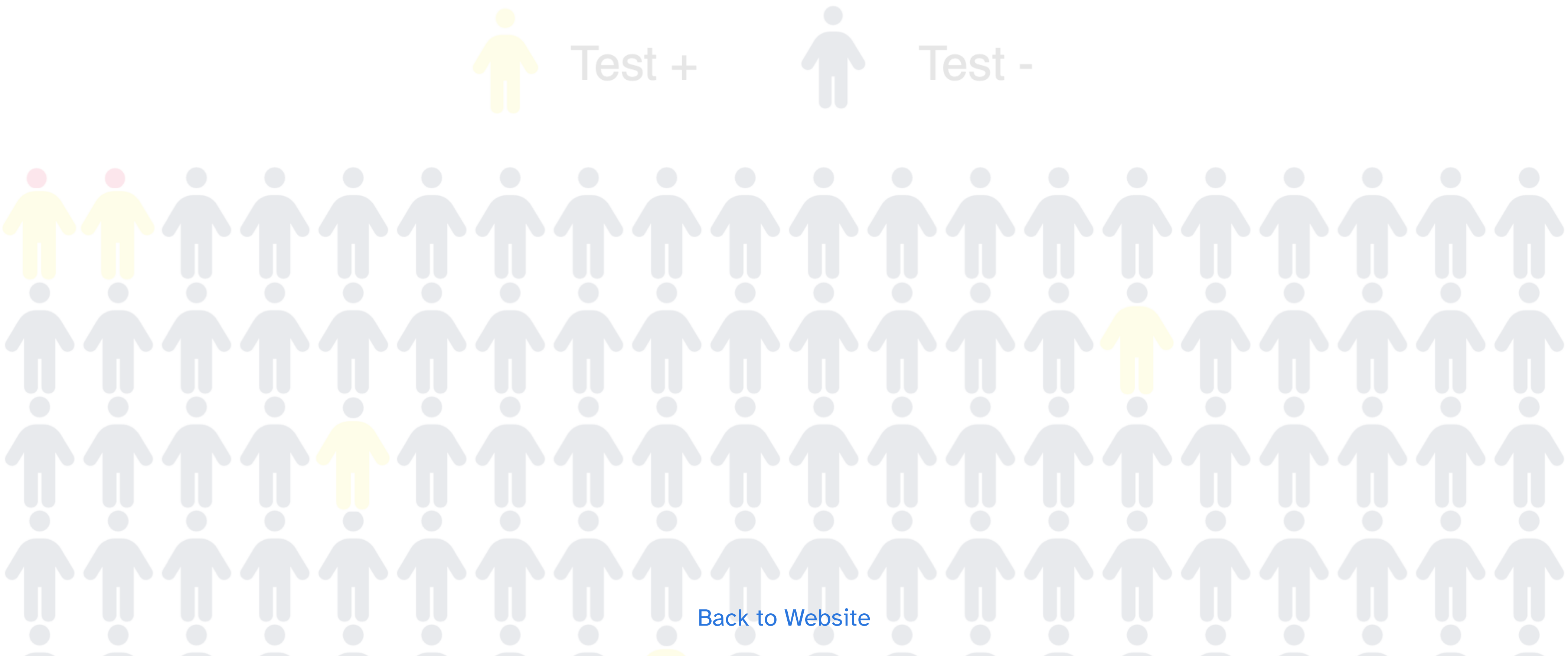

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&= \frac{2}{7.82}
\end{aligned}
$$

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2. 2x2 Tables

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2x2 Example: Disease Testing

Case example

You are trying to determine what proportion of the population has already been exposed a new communicable disease, in hopes of figuring out if herd immunity is possible.

You decide to do a antibody test to measure the level of antibodies in a sample of 500 participants

Disease Testing

Case example

What is the test’s SENSITIVITY?

What is the test’s SPECIFICITY?

What is the test’s FALSE NEGATIVE RATE?

What is the test’s FALSE POSITIVE RATE?

Disease Testing

Case example

	D+	*D-*	
T+	a (TP)	b (FP)	a + b
T-	c (FN)	d (TN)	c + d
	a + c	b + d	a + b + c + d

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Disease Testing: SENSITIVITY

	D+	*D-*	
T+	**125** (a, TP)	20 (b, FP)	145 (a + b)
T-	9 (c, FN)	346 (d, TN)	355 (c + d)
	134 (a + c)	366 (b + d)	500 (a + b + c + d)

Test Sensitivity among those who have or had the virus, 125/134 = 93% (Interpretation: The probability of the screening test correctly identifying diseased subjects was 93%)

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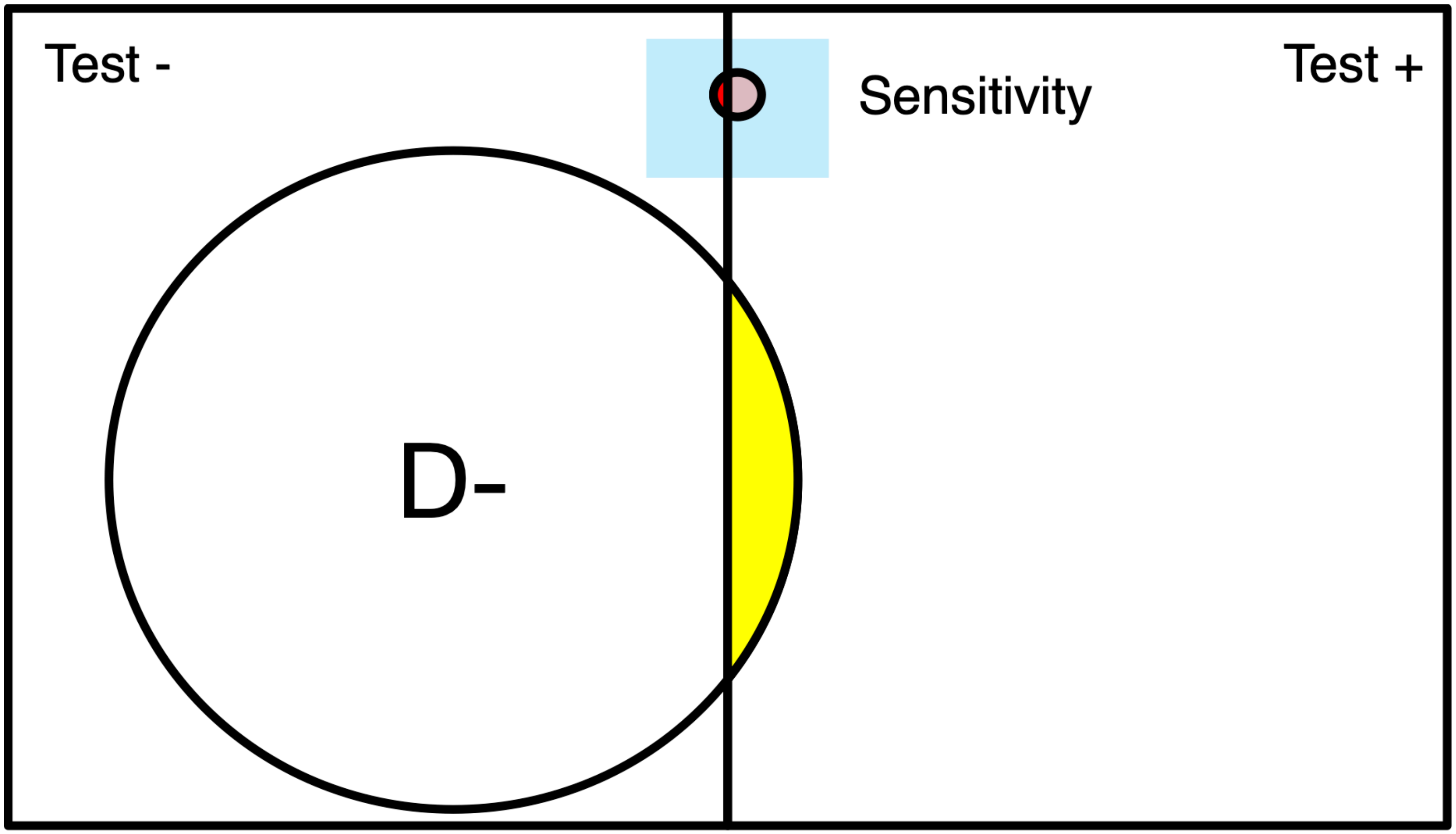
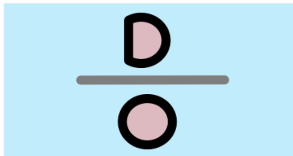
Disease Testing: SENSITIVITY



Sensitivity



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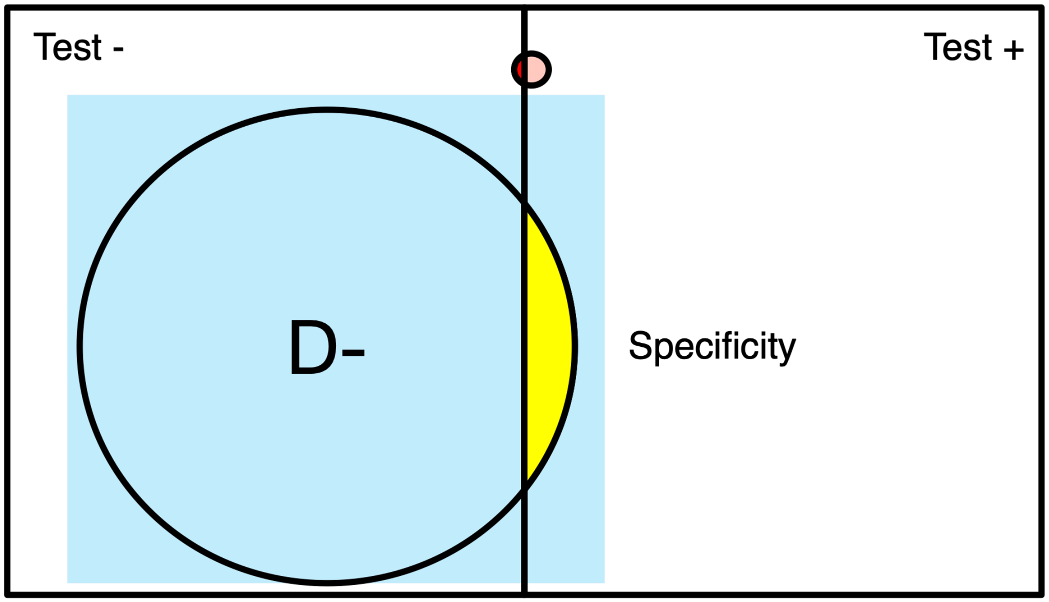
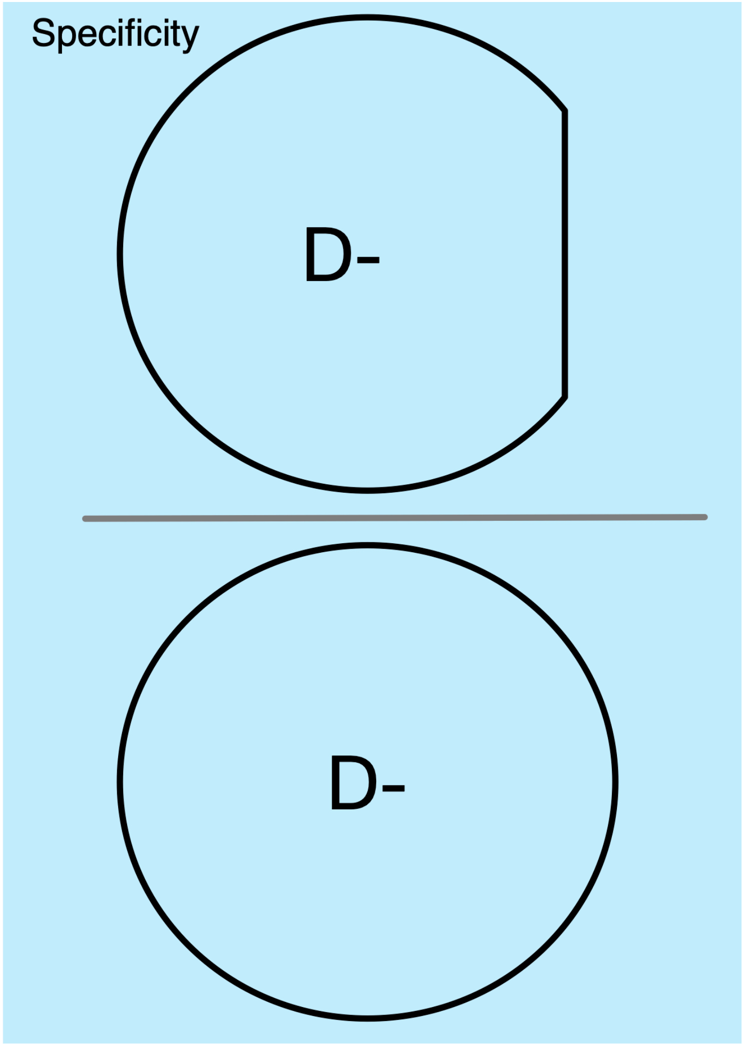
Example: SPECIFICITY

	D+	*D-*	
T+	125 (a, TP)	20 (b, FP)	145 (a + b)
T-	9 (c, FN)	**346** (d, TN)	355 (c + d)
	134 (a + c)	**366** (b + d)	500 (a + b + c + d)

Test Specificity among those without the disease at any point, 346/366 = 95% (Interpretation: The probability of the screening test correctly identifying non-diseased subjects was 65%)

Disease Testing: SPECIFICITY





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Do you want a test with good Sensitivity or good Specificity?

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False Negatives

False negative rate (1-sensitivity) is the proportion of diseased people with a negative test: c/(a+c)

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False Negatives

False negative rate (1-sensitivity) is the proportion of diseased people with a negative test: c/(a+c)

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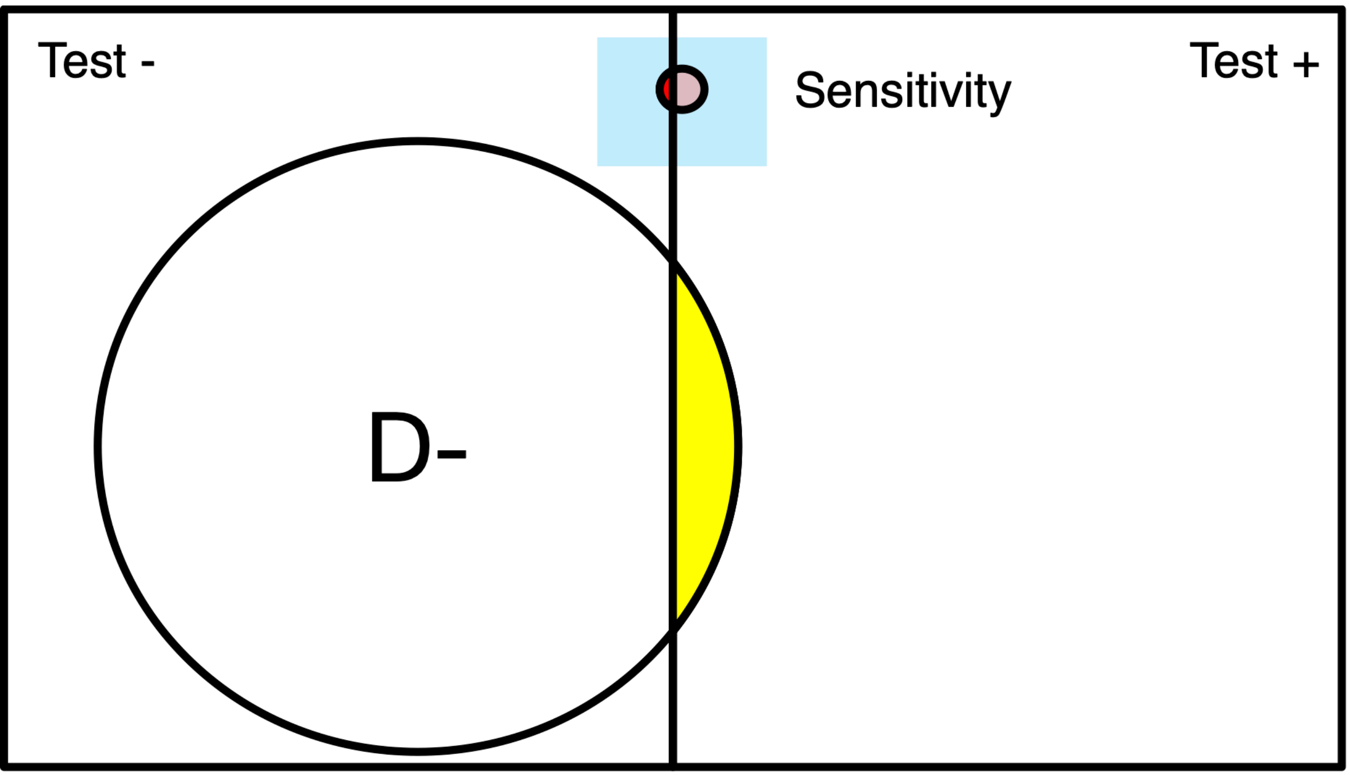
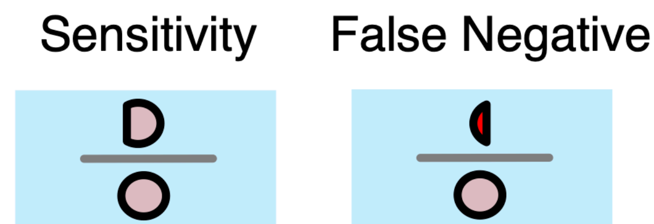
False Negatives





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False Positives

False positive rate (1-specificity) is the proportion of non-diseased people with a positive test: b/(b+d)

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False Negatives

False positive rate (1-specificity) is the proportion of non-diseased people with a positive test: b/(b+d)

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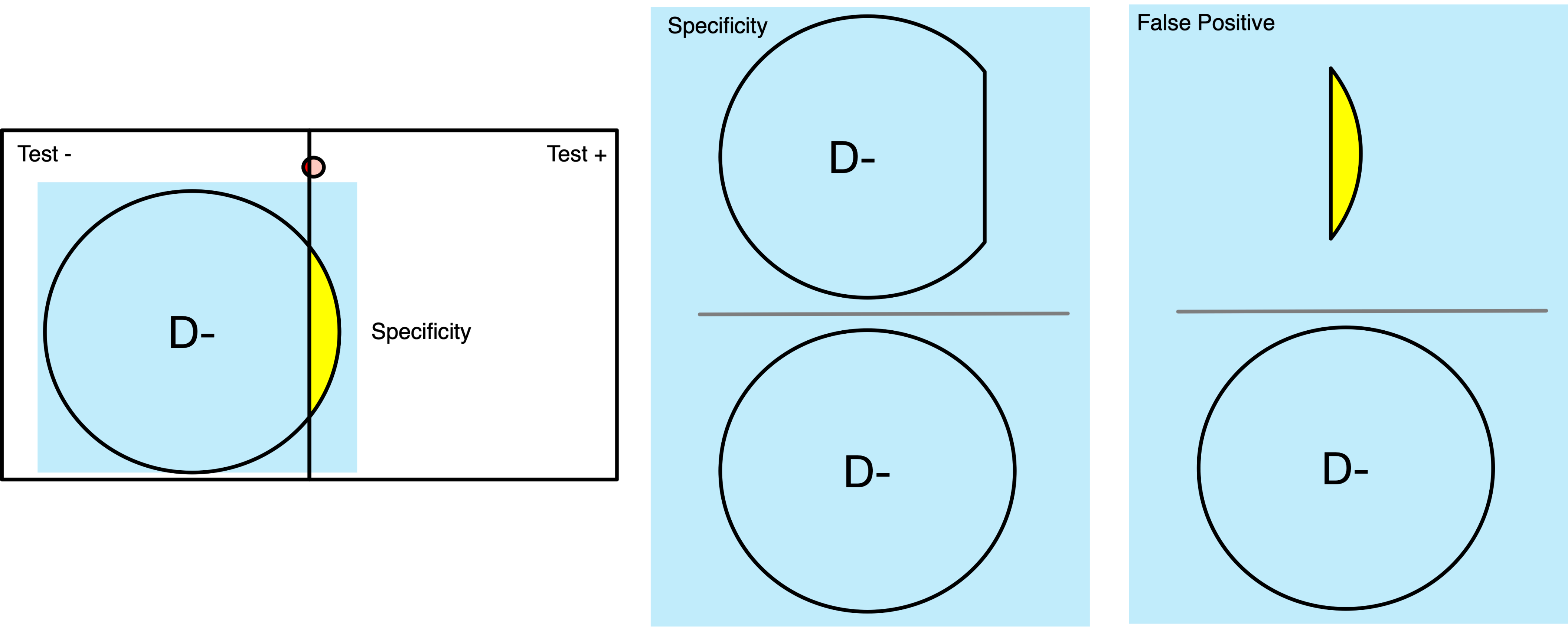
False Negatives



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Bayes' Theorem With Tree Inversion

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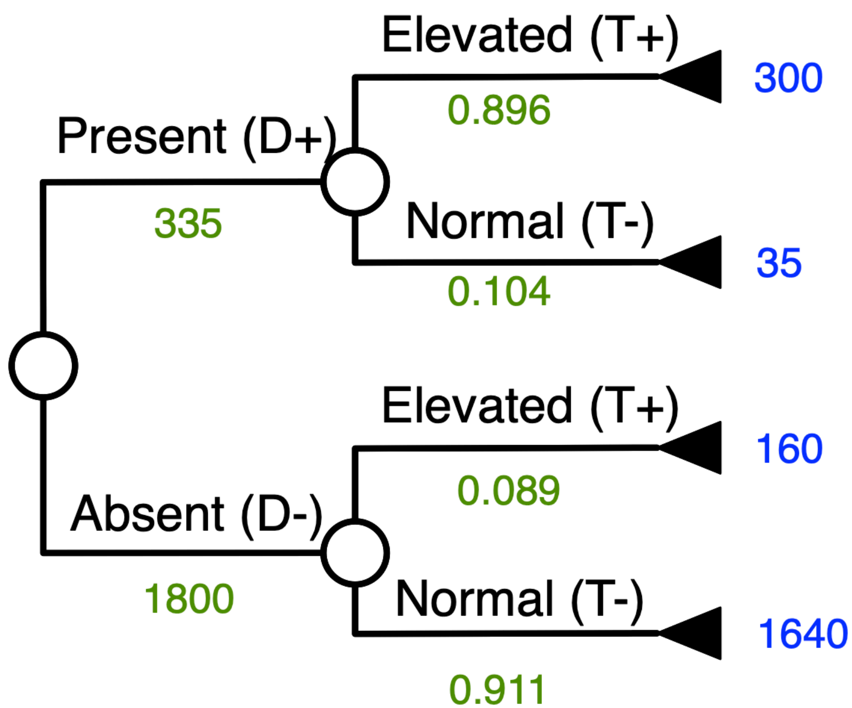
Bayes' theorem with tree inversion



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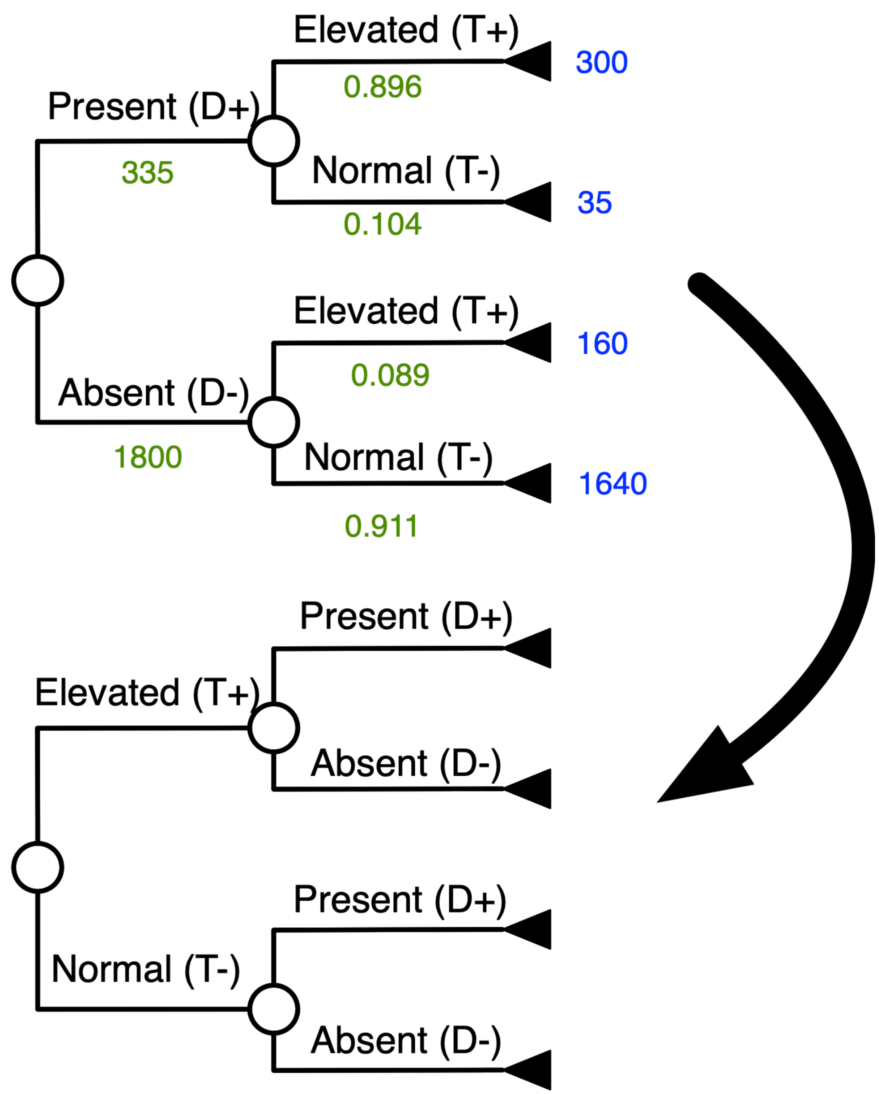
Goal: “Flip” The Tree



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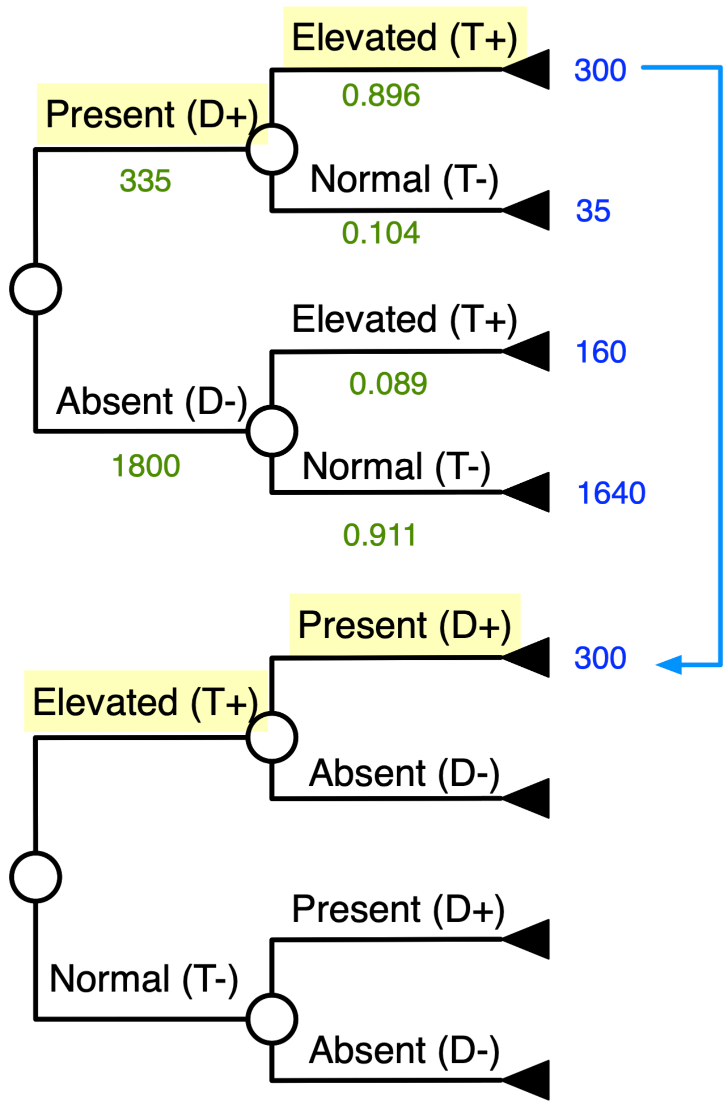
Start Matching Terminal Numbers …



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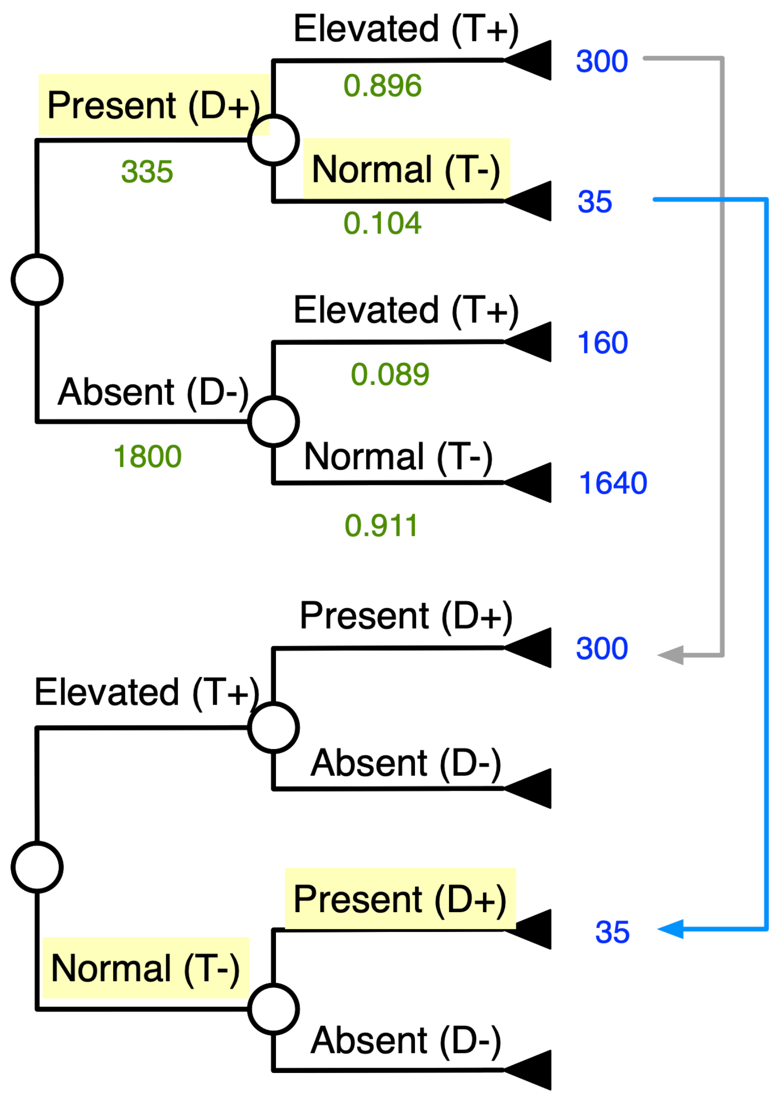
Continue Matching Terminal Numbers ...



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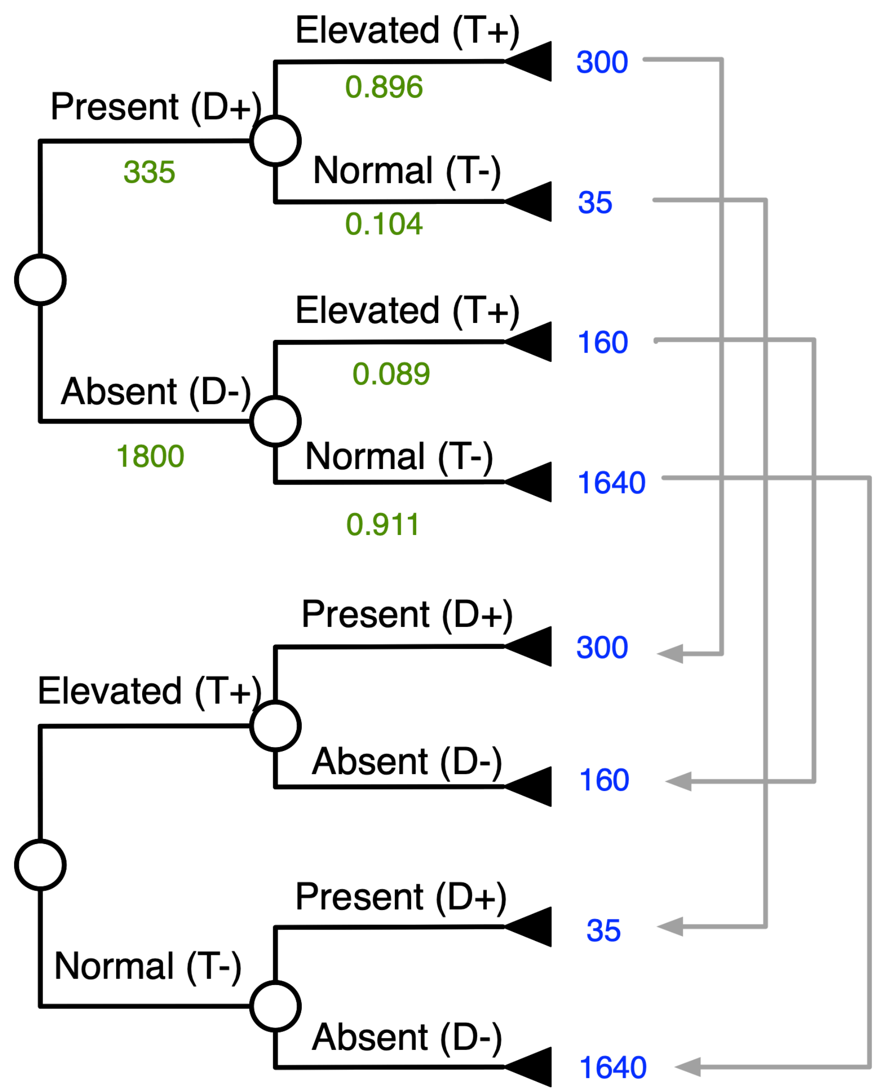
Continue Matching Terminal Numbers …



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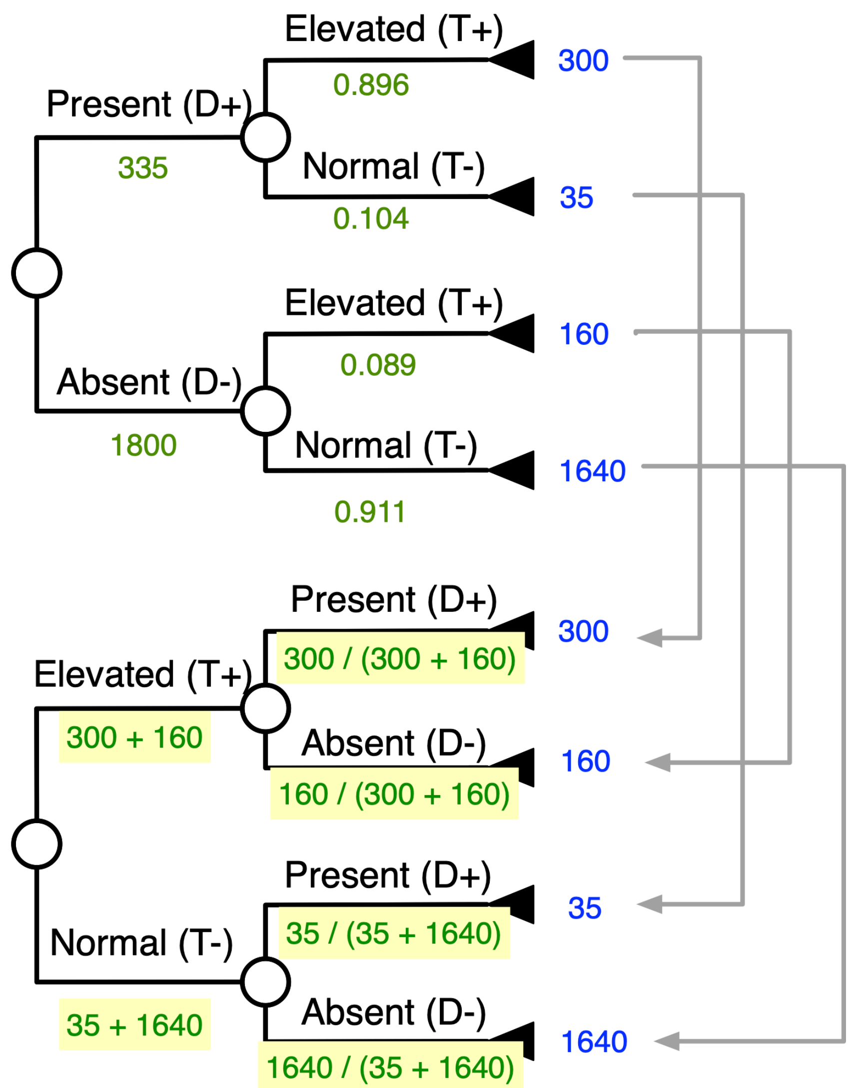
Work Your Way Down the Tree …



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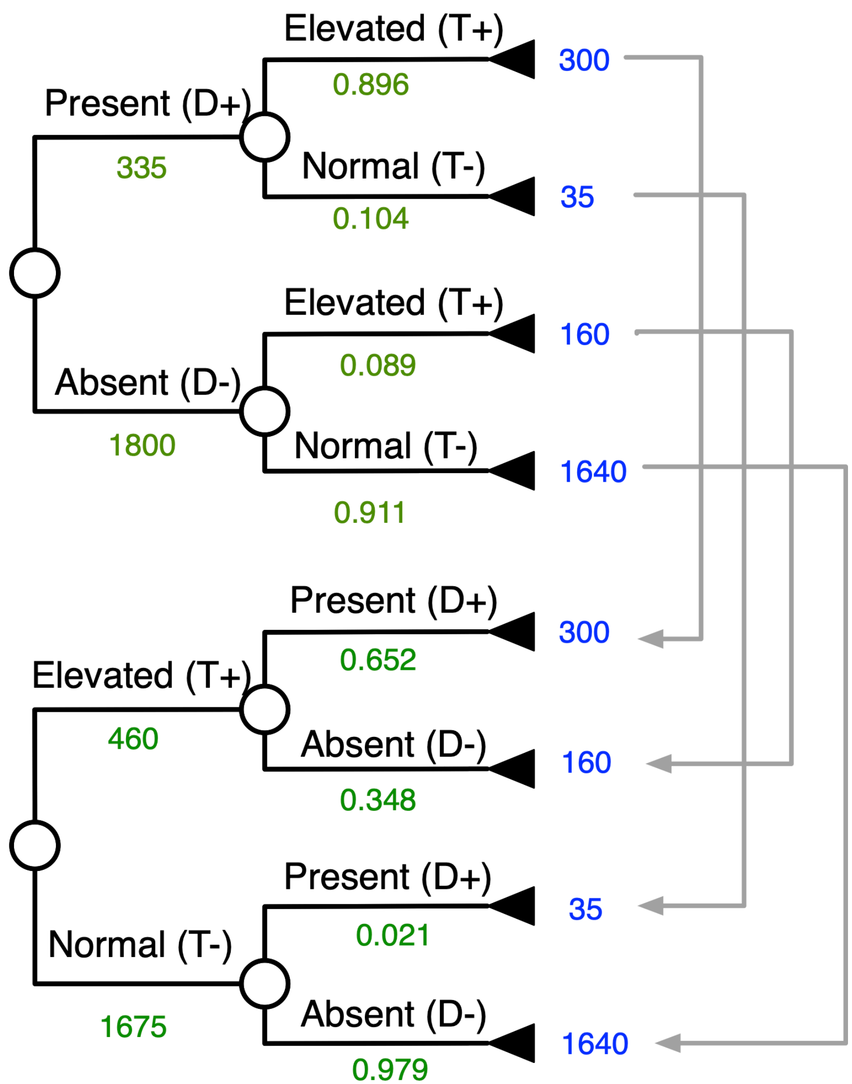
Inverted Tree



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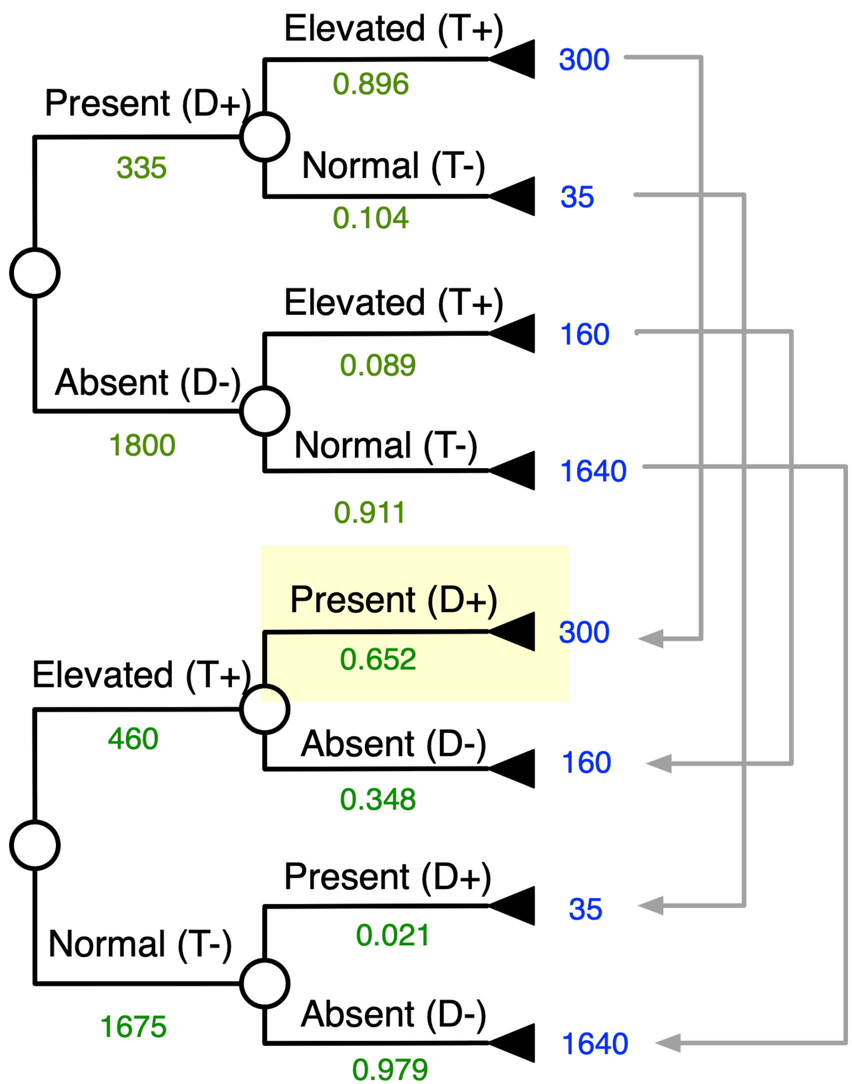
Probability of Disease Given + Test



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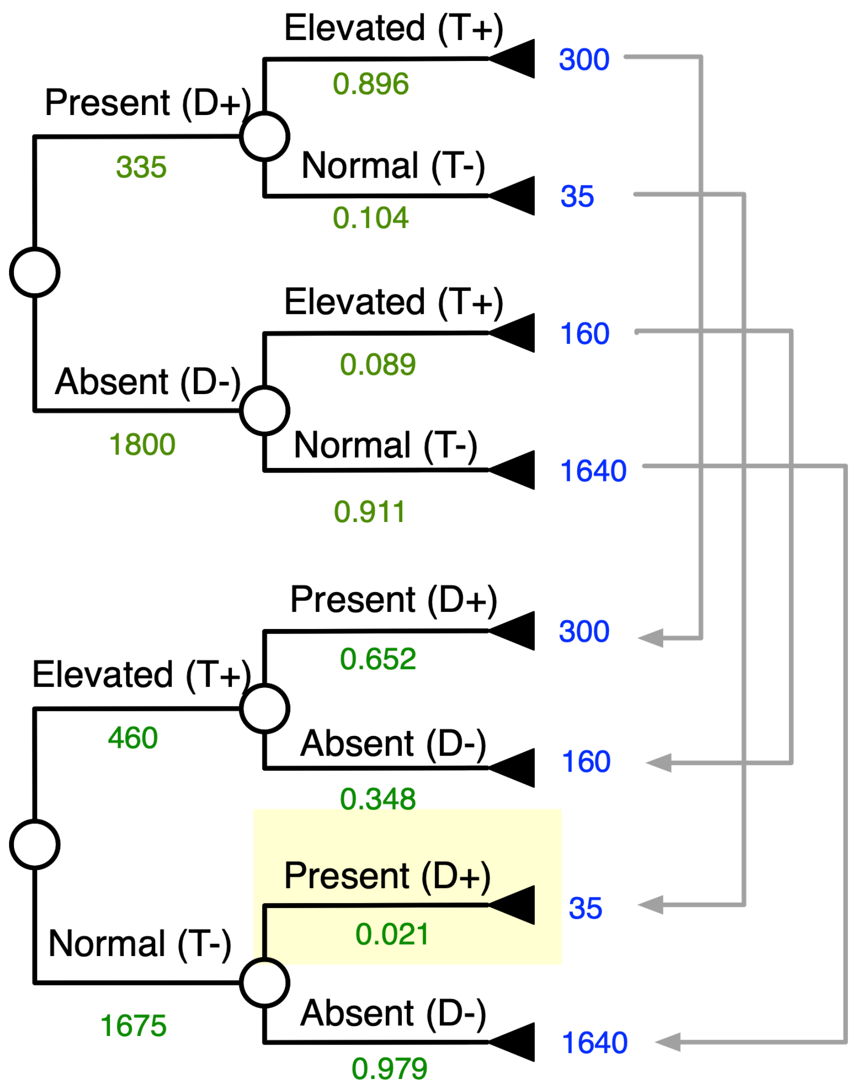
Probability of Disease Given - Test



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Conclusions

- We are often bayesian in how we think
- Knowledge of test characteristics can help you to make more informed decisions
- In general, diagnostic tests are helpful when pre-test prob is in the middle (30-70%) and test is going to move you past a decision threshold
- Tests with very good characteristics (ex. 100% sens or spec) can also be very useful if used appropriately

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[Extra] Likelihood Ratios

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Likelihood ratio

Useful for situations in which a quick estimate of revised probabilities is needed

> Likelihood that a given test result would be expected in a patient with the **target disorder Pr(test result | D+) compared to** the likelihood that the same result would be expected in a patient **without the target disorder Pr(test result | D-)** [A RATIO]

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Likelihood ratios

The likelihood ratio (LR) summarizes test sensitivity and specificity into one number:

LR (positive test) = sensitivity/1-specificity (or TPR/FPR)

LR (negative test) = 1-sensitivity/specificity (or FNR/TNR)

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Likelihood ratios

- LR can be used to revise disease probabilities using the following form of Bayes' Theorem:

> Post-test odds = Pretest odds x LR

- The odds that the patient has the target disorder, after the test results are known. It is calculated by multiplying the pre-test odds by the likelihood of a positive or negative test.

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Likelihood ratios

- LR’s are an advance beyond 2x2 tables
- To use likelihood ratios, you must be comfortable converting between probabilities of disease and odds of disease
- Odds are simply another way of describing the chances that something will (or won’t) happen

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Likelihood ratios

$$Odds of Disease = \frac{\text{Probability}}{\text{1 - Probability}}$$

$$Probability = \frac{\text{Odds}}{\text{Odds} + 1}$$

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Likelihood ratios

Odds favoring an event; Odds = p/(1-p) If an event has 0.20 probability of occurrence, the odds favoring the event = 0.2/0.8 = 0.25 (or 1:4)

Odds against (OddA) the event; OddA = (1-p)/p The odds against are 0.8/0.2 = 4 (or 4:1)

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Back to first example (coronary care unit)

	Present	*Absent*	
Elevated (+)	300 (a, TP)	15 (b, FP)	315 (a + b)
Normal (-)	35 (c, FN)	150 (d, TN)	185 (c + d)
	335 (a + c)	165 (b + d)	500 (a + b + c + d)

- Pre-test probability was 67% (prevalence)
- SENS = 0.90 & SPEC = 0.91
- LR+ (for positive test result) = SENS / 1=SPEC = 0.90/1-0/.91 = **10**
- Interpretation: A patient with a positive test result is 10X more likely to have had a heart attack than someone who did not have a heart attack with the same test result

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Back to first example (coronary care unit)

- LR - (for a negative test result):
  - 1-SENS / SPEC = 1-0.90 / 0.91 = **.11**, indicates a ~10-fold decrease in the odds of having a condition in a patient with a negative test result.

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Back to first example (coronary care unit)

Pre-test probability
Pre-test odds (0.67 / 1-0.67)
Post (+ test) odds of disease = pre-test odds * LR(+)
Post (+ test) prob of disease = post-test odds / post-test odds + 1
Post (- test) odds of disease = pre-test odds * LR(-)
Post (- test) prob of disease = 0.22 / 1.22

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Odds Likelihood Form of Bayes

$$OddsLR = \frac{\text{Pr(D+} \mid \text{test result)}}{\text{Pr(D-} \mid \text{test result)}} = \frac{Pr(D+)}{Pr(D-)} * \frac{lr(D+)}{lr(D-)}$$

$$\frac{Pr(D+)}{Pr(D-)} * \frac{lr(D+)}{lr(D-)}$$

The above is the same as:

$$\frac{\text{Pr(D+} \mid \text{test result)}}{\text{Pr(D-} \mid \text{test result)}}$$

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Odds Likelihood Form of Bayes

$$OddsLR = \frac{\text{Pr(D+ | test result)}}{\text{Pr(D- | test result)}} = \frac{Pr(D+)}{Pr(D-)} * \frac{\text{Pr(test result | D+)}}{\text{Pr (test result | D-)}}$$

Pre-test odds favoring disease (*the prior*):

$$\frac{Pr(D+)}{Pr(D-)}$$

The post-test odds given the test result:

$$\frac{\text{Pr(D+ | test result)}}{\text{Pr(D- | test result)}}$$

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Odds Likelihood Form of Bayes

$$\frac{\text{Pr(test result} \mid \text{D+)}}{\text{Pr (test result} \mid \text{D-)}} = \frac{Pr(D-)}{Pr(D+)} * \frac{(CTN - CFP)}{(CTP - CFN)}$$

How to calculate an optimal "cut-off" for a test with categorical or continuous results at the point in which we will optimize the cut-off conditional on the (1) prior probability of disease and (2) the consequences of the scenario we are assessing

Next lecture: Positivity Criterion!

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Likelihood ratios

LR (+)
GT 10 Excellent
5-10 Good
2-5 Fair. May be helpful
1-2 Unlikely to be helpful

LR (-)
<0.1 Excellent
0.1-0.2 Good
0.2-0.5 Fair. May be helpful
0.5-1.0 Unlikely to be helpful

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